Path-dependent equations and viscosity solutions in infinite dimension

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Abstract

Path Dependent PDE's (PPDE's) are natural objects to study when one deals with non Markovian models. Recently, after the introduction (see [12]) of the so-called pathwise (or functional or Dupire) calculus, various papers have been devoted to study the well-posedness of such kind of equations, both from the point of view of regular solutions (see e.g. [18]) and viscosity solutions (see e.g. [13]), in the case of finite dimensional underlying space. In this paper, motivated by the study of models driven by path dependent stochastic PDE's, we give a first well-posedness result for viscosity solutions of PPDE's when the underlying space is an infinite dimensional Hilbert space. The proof requires a substantial modification of the approach followed in the finite dimensional case. We also observe that, differently from the finite dimensional case, our well-posedness result, even in the Markovian case, apply to equations which cannot be treated, up to now, with the known theory of viscosity solutions.

Key words: Viscosity solutions, path-dependent stochastic differential equations, path-dependent partial differential equations, partial differential equations in infinite dimension.

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1 Introduction

Given $T > 0$ and a real separable Hilbert space H, let $C([0,T];H)$ be the Banach space of continuous functions from $[0,T]$ to H, endowed with the supremum norm $\|\mathbf{x}\|_\infty := \sup_{t\in[0,T]} |\mathbf{x}(t)|$, for all $\mathbf{x} \in C([0,T];H)$. Let $\Lambda := [0,T] \times C([0,T];H)$ and consider the following pseudometric on Λ:

$$\mathbf{d}_\infty\big((t,\mathbf{x}),(t',\mathbf{x}')\big) \ := \ |t-t'| + \|\mathbf{x}_{\cdot\wedge t} - \mathbf{x}'_{\cdot\wedge t'}\|_\infty, \qquad (t,\mathbf{x}),\ (t',\mathbf{x}') \in \Lambda.$$

This pseudo-metric allows to account for the *non-anticipativity* condition: each function $v : (\Lambda, \mathbf{d}_\infty) \to E$, where E is a Banach space, which is measurable with respect to the Borel σ-algebra induced by $\mathbf{d}_\infty$, is such that $v(t,\mathbf{x}) = v(t,\mathbf{x}_{\cdot\wedge t})$ for all $(t,\mathbf{x}) \in \Lambda$. Let A be the generator of a strongly continuous semigroup on H, and let $b\colon \Lambda \to H$, $\sigma\colon \Lambda \to L(K;H)$, where K is another real separable Hilbert space (the noise space, as we will see in Section 3). In this paper, we study the wellposedness of the following infinite dimensional semilinear path-dependent partial differential equation (PPDE):

$$-\partial_t u - \langle A\mathbf{x}_t, \partial_\mathbf{x} u\rangle - \langle b(t,\mathbf{x}), \partial_\mathbf{x} u\rangle - \frac{1}{2}\mathrm{Tr}\big[\sigma(t,\mathbf{x})\sigma^*(t,\mathbf{x})\partial^2_{\mathbf{x}\mathbf{x}} u\big] - F(t,\mathbf{x},u) \ = \ 0, \quad (1.1)$$

for all $t \in [0,T)$ and $\mathbf{x} \in C([0,T];H)$, where $F\colon \Lambda \times \mathbb{R} \to \mathbb{R}$ and $\partial_t u$, $\partial_\mathbf{x} u$, $\partial^2_{\mathbf{x}\mathbf{x}} u$ denote the so-called pathwise (or functional or Dupire, see [5, 6, 12]) derivatives. The unknown is a non-anticipative functional $u\colon \Lambda \to \mathbb{R}$. We are deliberately restricting the nonlinearity F to depend only on u, and not on $\partial_\mathbf{x} u$, in order to focus on our main wellposedness objective. More general nonlinearities are left for future research.

In addition to the infinite dimensional feature of the equation (1.1), we emphasize that its coefficients b, σ, and F are path-dependent. Such a path-dependency may be addressed the standard PDE approach, by introducing a "second level" of infinite-dimensionality, embedding the state space H in a larger infinite-dimensional space like e.g. $L^2(-T,0;H)$, and converting equation (1.1) into a PDE on this larger space (see e.g., in the context of *delay equations* and when the original space H is finite-dimensional, [3, 9, 16]). The latter methodology turns out to be problematic when the data, as in our case, are required to have continuity properties with respect to the supremum norm, as the PDE should be considered basically in spaces of continuous functions, which are not reflexive. However, we should mention that some attempts have been achieved along this direction, we refer to [10, 11, 17, 18, 19].

When the space H is finite-dimensional, PPDEs with a structure more general than (1.1) have been investigated by means of a new concept of viscosity solution recently introduced in [13], and further developed in [14, 15, 33]. This new notion enlarges the class of test functions, by defining the smoothness only "with respect to the dynamics" of the underlying stochastic system and requiring the usual "tangency condition" - required locally pointwise in the standard viscosity definition - only in mean. These two weakenings, on one hand, keep safe the existence of solutions; on the other hand, simplify a lot the proof of uniqueness - as it does not require anymore the passage through the Crandall-Ishii Lemma.

The main objective of this paper is to extend to our infinite-dimensional path-dependent context such new notion of viscosity solution. Before illustrating our results, we recall that, for equation like (1.1), when all coefficient are Markovian, results on existence and uniqueness of classical solutions (that can be found e.g. in [9, Chapter 7]) are much weaker than in the finite dimensional case, due to the lack of local compactness and to the absence of a reference measure like the Lebesgue one. This makes quite relevant the notion of viscosity solution, introduced in the infinite-dimensional case by [26, 27, 28], see also [35] and, for a survey, [16, Chapter 3]. The infinite dimensional extension of the usual notion of viscosity solution to these PDEs is not trivial, as the comparison results are established only under strong continuity assumptions on the coefficients (needed to generate maxima and minima) and under a nuclearity condition on the diffusion coefficient σ. The latter purely technical condition is a methodological bound of this notion of viscosity solutions, as it is only needed in order to adapt the Crandall-Ishii Lemma to the infinite-dimensional context.

The core results of the present paper (contained in the main Section 4) are as follows. First, on the line of [33], we show that the infinite-dimensional definition has an equivalent version with semijets (Proposition 3.6). Then, under natural assumptions on the operator A and the coefficients b, σ, F, we prove sub/supermartingale characterization of sub/supersolutions which extends the corresponding result in [33] (Theorem 4.8). As a corollary of this characterization we get that the PPDE satisfies the desired stability property of viscosity solutions (Proposition 4.13). Furthermore still applying Theorem 4.8 we prove that equation (1.1) satisfies the comparison principle in the class of continuous functions with polynomial growth on Λ (Corollary 4.15). In particular, since the Crandall-Ishii Lemma is not needed to establish comparison, we emphasize that the nuclearity condition on σ is completely by-passed in our framework. Similarly this happens for the strong continuity properties mentioned above. Finally, given a uniformly continuous terminal condition $u(T, \mathbf{x}) = \xi(\mathbf{x})$, we establish existence of a unique solution (Theorem 4.17). We observe that our unique viscosity solution is closely related to the solution of the infinite dimensional backward stochastic differential equation (BSDE) of [20], which can be viewed as a Sobolev solution to equation (1.1) (see e.g. [2]).

From what we have said, it follows that the passage from finite to infinite dimension makes meaningful considering the new notion of viscosity solution *even* in the Markovian (no path-dependent case). Indeed, while in the finite dimensional case the theory based on the usual definition of viscosity solutions is so well-developed to cover basically a huge class of PDEs, in the infinite dimensional case the known theory of viscosity solutions collides with the structural constraints described above, which can be by-passed with the new notion.

Finally, we point out that our results may be extended to suitable nonlinearities depending on the gradient $\partial_{\mathbf{x}} u$. In our formalism, a way to do it could be by introducing a control process in the drift of the underlying stochastic system, which basically corresponds, in the formalism [13], to replace the expectation in the tangency condition on test function by a nonlinear expectation operator defined as sup/inf of expectations under a convenient family of probability measures. This paper deliberately avoids this additional complication in order to focus on the infinite-dimensional feature of the equations, and we leave the study of more general nonlinearities to future work.

This paper is organized as follows. In Section 2 we present the notation used throughout the paper. Section 3 is devoted to the study of existence, uniqueness, and stability of mild solutions of path-dependent SDEs in Hilbert spaces, presenting a result which is not contained in the current literature. In Section 4 we introduce the notion of viscosity solution for path-dependent PDEs in Hilbert spaces, in terms of both test functions and semijets (Subsection 4.1); we prove a martingale characterization of viscosity sub/supersolutions and a stability result (Subsection 4.2); finally, we prove the comparison principle (Subsection 4.3) and we provide an existence and uniqueness result for the path-dependent PDE (Subsection 4.4). In the last section, Section 5, we consider the Markovian case, i.e., when all data depend only on the present, and we compare the notion of viscosity solution studied in Section 4 to the usual notions of viscosity solutions adopted in the literature for partial differential equations in Hilbert spaces. Finally the Appendix is devoted to a clarification on the definition of test functions given in Subsection 4.1.

2 Notation

Consider a real separable Hilbert space H. Denote by $\langle\cdot,\cdot\rangle$ and $|\cdot|$ the scalar product and norm on H, respectively. Let $T>0$ and consider the Banach space

$$\mathbb{W} := C([0,T];H)$$

of continuous functions from $[0,T]$ to H, whose generic element is denoted by $\mathbf{x}$ and whose norm is denoted by $\|\cdot\|_\infty$, i.e., $\|\mathbf{x}\|_\infty := \sup_{t\in[0,T]}|\mathbf{x}_t|$. Introduce the space

$$\Lambda := [0,T]\times\mathbb{W}$$

and the map $\mathbf{d}_\infty\colon \Lambda\times\Lambda\to\mathbb{R}^+$ defined by ([1])

$$\mathbf{d}_\infty\big((t,\mathbf{x}),(t',\mathbf{x}')\big) \;:=\; |t-t'| + \|\mathbf{x}_{\cdot\wedge t}-\mathbf{x}'_{\cdot\wedge t'}\|_\infty.$$

Then $\mathbf{d}_\infty$ is a pseudometric on Λ. In particular, $(\Lambda,\mathbf{d}_\infty)$ is a topological space with the topology induced by the pseudometric $\mathbf{d}_\infty$. The quotient space $(\Lambda\,/\sim)$, where $\sim$ is the equivalence relation defined by

$$(t,\mathbf{x})\sim(t',\mathbf{x}') \qquad \text{whenever} \qquad t=t',\;\; \mathbf{x}_s=\mathbf{x}'_s\;\;\forall\, s\in[0,t],$$

is a complete separable metric space when endowed with the quotient metric. Λ becomes a measurable space when endowed with the Borel σ-algebra induced by $\mathbf{d}_\infty$. Throughout the paper, the topology and σ-algebra on Λ are those induced by $\mathbf{d}_\infty$.

Definition 2.1. *Let E be a Banach space. An E-valued* ***non-anticipative functional*** *on Λ is a map $v\colon\Lambda\to E$ such that*

$$v(t,\mathbf{x}) = v(t,\mathbf{x}_{\cdot\wedge t}), \qquad \forall\,(t,\mathbf{x})\in\Lambda.$$

[1] We use the same symbol $|\cdot|$ to denote both the *norm* on H and the *absolute value* of a real number. However no confusion should arise, since the real meaning will be clear from the context.

Definition 2.2. *Let $(E, |\cdot|_E)$ be a Banach space.*

(i) *$C(\Lambda; E)$ is the space of continuous functions $v\colon \Lambda \to E$.*

(ii) *$C_p(\Lambda; E)$, $p \geq 1$, is the space of continuous functions $v\colon \Lambda \to E$ satisfying the following polynomial growth condition :*

$$|v(t, \mathbf{x})|_E \;\leq\; M(1 + \|\mathbf{x}\|_\infty^p), \qquad \forall\,(t, \mathbf{x}) \in \Lambda,$$

for some constant $M > 0$. $C_p(\Lambda; E)$ is a Banach space when endowed with the norm

$$|v|_{C_p(\Lambda;E)} \;:=\; \sup_{(t,\mathbf{x})\in\Lambda} \frac{|v(t,\mathbf{x})|_E}{(1 + \|\mathbf{x}\|_\infty)^p}.$$

(iii) *$UC(\Lambda, E)$ is the space of uniformly continuous functions $v\colon \Lambda \to E$.*

When $E = \mathbb{R}$, we drop $\mathbb{R}$ and simply write $C(\Lambda)$, $C_p(\Lambda)$, and $UC(\Lambda)$.

Remark 2.3. **(1)** *Clearly, for all $p \geq 1$, we have the inclusions $UC(\Lambda, E) \subset C_1(\Lambda, E) \subset C_p(\Lambda, E) \subset C(\Lambda, E)$.*

(2) *A measurable map $v\colon \Lambda \to E$ is automatically non-anticipative. For this reason, we will drop the term non-anticipative when v is measurable.* □

Now let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions. We shall make use of the following classes of stochastic processes on this space.

Definition 2.4. *Let $(E, |\cdot|_E)$ be a Banach space.*

(i) *$L^0_{\mathcal{P}}(E) := L^0_{\mathcal{P}}(\Omega \times [0,T]; E)$ is the space ([2]) of E-valued predictable processes X, endowed with the topology induced by the convergence in measure.*

(ii) *$L^p_{\mathcal{P}}(E) := L^p_{\mathcal{P}}(\Omega \times [0,T]; E)$, $p \geq 1$, is the Banach space of E-valued predictable processes X such that*

$$\|X\|^p_{L^p_{\mathcal{P}}(E)} \;:=\; \mathbb{E}\left[\int_0^T |X_t|_E^p dt\right] \;<\; \infty.$$

(iii) *$\mathcal{H}^0_{\mathcal{P}}(E)$ is the subspace of elements $X \in L^0_{\mathcal{P}}(E)$ admitting a continuous version. Given an element of $\mathcal{H}^0_{\mathcal{P}}(E)$ we shall always refer to its uniquely determined (up to a $\mathbb{P}$-null set) continuous version.*

(iv) *$\mathcal{H}^p_{\mathcal{P}}(E)$, $p \geq 1$, is the subspace of elements $X \in L^p_{\mathcal{P}}(E)$ admitting a continuous version and such that*

$$\|X\|^p_{\mathcal{H}^p_{\mathcal{P}}(E)} \;:=\; \mathbb{E}\Big[\sup_{t\in[0,T]} |X_t|_E^p\Big] \;<\; \infty.$$

$\mathcal{H}^p_{\mathcal{P}}(E)$, when endowed with the norm $\|\cdot\|_{\mathcal{H}^p_{\mathcal{P}}(E)}$ defined above, is a Banach space.

When $E = \mathbb{R}$, we drop $\mathbb{R}$ and simply write $L^0_{\mathcal{P}}$, $L^p_{\mathcal{P}}$, $\mathcal{H}^0_{\mathcal{P}}$, and $\mathcal{H}^p_{\mathcal{P}}$.

[2]The subscript $\mathcal{P}$ in $L^0_{\mathcal{P}}(E)$, and in the other spaces introduced in Definition 2.4, refers to the adjective *predictable*.

Remark 2.5. *In the present paper, as it is usually done in the literature on infinite dimensional second order PDEs (see, e.g., [8, 35]), we distinguish between the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, whose generic element is ω, and the path space $\mathbb{W}$, whose generic element is $\mathbf{x}$. Instead, in [13], the authors identify these two spaces (up to the translation of the initial point), taking as probability space the canonical space $\{\mathbf{x} \in \mathbb{W} \colon \mathbf{x}_0 = 0\}$ and calling ω its generic element. Clearly everything done here can be rephrased in the setting of [13] (again up to a translation of the initial point), by taking as probability space $(\mathbb{W}, \mathcal{B}(\mathbb{W}), \mathbb{P}^X)$, where $\mathcal{B}(\mathbb{W})$ is the σ-algebra of Borel subsets of $\mathbb{W}$ and $\mathbb{P}^X$ is the law of the process X that we shall define in the next section as mild solution of a path-dependent SDE.* $\square$

3 Path-dependent SDEs in Hilbert spaces

In this section we define and study a path-dependent SDE in Hilbert space. As general references for stochastic integration and SDEs in infinite-dimensional spaces, we refer to the monographies [8, 21].

Let K be a real separable Hilbert space and let $W = (W_t)_{t\geq 0}$ be a K-valued cylindrical Wiener process on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We consider, for $t \in [0,T]$ and $Z \in \mathcal{H}^0_{\mathcal{P}}(H)$, the following *path-dependent* SDE:

$$\begin{cases} dX_s = AX_s ds + b(s, X)ds + \sigma(s, X)dW_s, & s \in [t,T], \\ X_{\cdot\wedge t} = Z_{\cdot\wedge t}. \end{cases} \tag{3.1}$$

The precise notion of solution is given below. First, we introduce some notations and then impose Assumption 3.1 on A, b, σ. We denote by $L(K;H)$ the Banach space of bounded linear operators from K to H, endowed with the operator norm. We also denote by $L_2(K;H)$ the Hilbert space of Hilbert-Schmidt operators from K to H, whose scalar product and norm are, respectively,

$$\langle P, Q\rangle_{L_2(K;H)} \;:=\; \sum_{k=1}^{\infty} \langle Pe_k, Qe_k\rangle, \qquad \|P\|_{L_2(K;H)} \;:=\; \left(\sum_{k=1}^{\infty} |Pe_k|^2\right)^{1/2},$$

for all $P, Q \in L_2(K;H)$, where $\{e_k\}_k$ is a complete orthonormal basis of K ([3]).

Assumption 3.1.

(i) *The operator $A \colon \mathcal{D}(A) \subset H \to H$ is the generator of a strongly continuous semigroup $\{e^{tA},\, t \geq 0\}$ in the Hilbert space H.*

(ii) *$b \colon \Lambda \to H$ is measurable and such that, for some constant $M > 0$,*

$$|b(t, \mathbf{x}) - b(t, \mathbf{x}')| \;\leq\; M\|\mathbf{x} - \mathbf{x}'\|_\infty, \qquad |b(t, \mathbf{x})| \;\leq\; M(1 + \|\mathbf{x}\|_\infty),$$

for all $\mathbf{x}, \mathbf{x}' \in \mathbb{W}$, $t \in [0,T]$.

[3] We recall that, for any $P, Q \in L_2(K;H)$, the quantities $\langle P, Q\rangle_{L_2(K;H)}$ and $\|P\|_{L_2(K;H)}$ are independent of the choice of the basis $\{e_k\}_k$. Moreover, we recall that $L_2(K;H)$ is separable, as every operator P in $L_2(K;H)$ is compact.

(iii) $\sigma\colon \Lambda \to L(K;H)$ *is such that* $\sigma(\cdot,\cdot)v\colon \Lambda \to H$ *is measurable for each* $v \in K$ *and* $e^{sA}\sigma(t,\mathbf{x}) \in L_2(K;H)$ *for every* $s > 0$ *and every* $(t,\mathbf{x}) \in \Lambda$. *Moreover, there exist* $\hat{M} > 0$ *and* $\gamma \in [0,1/2)$ *such that, for all* $\mathbf{x}, \mathbf{x}' \in \mathbb{W}$, $t \in [0,T]$, $s \in (0,T]$,

$$\|e^{sA}\sigma(t,\mathbf{x})\|_{L_2(K;H)} \;\leq\; \hat{M}s^{-\gamma}(1+\|\mathbf{x}\|_\infty), \tag{3.2}$$

$$\|e^{sA}\sigma(t,\mathbf{x}) - e^{sA}\sigma(t,\mathbf{x}')\|_{L_2(K;H)} \;\leq\; \hat{M}s^{-\gamma}\|\mathbf{x}-\mathbf{x}'\|_\infty. \tag{3.3}$$

Remark 3.2. *1. Regarding Assumption 3.1(iii), we observe that one could do the more demanding assumption of sublinear growth and Lipschitz continuity of* $\sigma(t,\cdot)$ *as function valued in the space* $L_2(K;H)$ *(see [21]). The assumption we give, which is the minimal one used in literature to give sense to the stochastic integral and to ensure the continuity of the stochastic convolution, is taken from [8, Hypothesis 7.2] and [20].*

2. Regarding Assumption 3.1(ii), we observe that it could be relaxed giving assumptions on the composition of the map b *with the semigroup, as done for* σ *in part (iii) of the same Assumption. Here, we follow [8, 20] and we do not perform it.*

Before giving the precise notion of solution to (3.1) we make some observations.

(O1) For $p = 0$ and $p \geq 1$, we have the isometric embedding ([4])

$$\mathcal{H}^p_{\mathcal{P}}(H) \;\hookrightarrow\; L^p(\Omega,\mathcal{F},\mathbb{P};\mathbb{W}).$$

Hence a process in $\mathcal{H}^p_{\mathcal{P}}(H)$, $p = 0$ or $p \geq 1$, can be seen (and we shall adopt this point of view in many points throughout the paper) as an $\mathbb{W}$-valued random variable.

(O2) If $X \in \mathcal{H}^p_{\mathcal{P}}(H)$, $p = 0$ or $p \geq 1$, then $X_{\cdot\wedge t} \in L^p(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{W})$.

(O3) We have the continuous inclusion (denoting $\mathbf{d}((t,\mathbf{x}),(s,\mathbf{y})) = |t-s| + \|\mathbf{x}-\mathbf{y}\|_\infty$, $\forall\,(t,\mathbf{x}),(s,\mathbf{y}) \in \Lambda$, the standard metric on Λ)

$$(\Lambda,\mathbf{d}) \;\hookrightarrow\; (\Lambda,\mathbf{d}_\infty),$$

due to the inequality

$$\mathbf{d}_\infty((t,\mathbf{x}),(s,\mathbf{y})) \;\leq\; |t-s| + (w_{\mathbf{x}} \wedge w_{\mathbf{y}})(|t-s|) + \|\mathbf{x}-\mathbf{y}\|_\infty, \quad \forall\,(t,\mathbf{x}),\ (s,\mathbf{y}) \in \Lambda,$$

where $w_{\mathbf{x}}$, $w_{\mathbf{y}}$ are moduli of continuity of $\mathbf{x},\mathbf{y}$, respectively.

(O4) Given $v \in C(\Lambda,H)$ and $X \in \mathcal{H}^0_{\mathcal{P}}(H)$, due to (O1)–(O3) above, the composition $v(\cdot,X)$ belongs to $\mathcal{H}^0_{\mathcal{P}}(H)$.

(O5) Given $v \in C_q(\Lambda,H)$ and $X \in \mathcal{H}^p_{\mathcal{P}}(H)$, with $1 \leq q \leq p < \infty$, due to (O1)–(O3) above, the composition $v(\cdot,X)$ is a process in the class $\mathcal{H}^{p/q}_{\mathcal{P}}(H)$.

Definition 3.3. *Let* $Z \in \mathcal{H}^0_{\mathcal{P}}(H)$. *We call* ***mild solution*** *of* (3.1) *a process* $X \in \mathcal{H}^0_{\mathcal{P}}(H)$ *such that* $X_{\cdot\wedge t} = Z_{\cdot\wedge t}$ *and*

$$X_s \;=\; e^{(s-t)A}Z_t + \int_t^s e^{(s-r)A}b(r,X)dr + \int_t^s e^{(s-r)A}\sigma(r,X)dW_r, \quad \forall\, s \in [t,T]. \tag{3.4}$$

[4] In the case $p = 0$, the spaces $\mathcal{H}^0_{\mathcal{P}}$ and $L^0_{\mathcal{P}}$ are endowed with the metrics associated to the convergence in measure (see [30, Ch. 1, Sec. 5]).

Remark 3.4. *The condition* (3.2) *implies*

$$\int_t^s \|e^{(s-r)A}\sigma(r,\mathbf{x})\|^2_{L_2(K;H)}dr \ \leq \ C_0(1+\|\mathbf{x}\|^2_\infty), \quad \forall\, 0\leq t\leq s\leq T,\ \forall\,\mathbf{x}\in\mathbb{W},$$

which ensures that the stochastic integral in Definition 3.3 makes sense for every process $X\in\mathcal{H}^0_{\mathcal{P}}(H)$.

We are going to state an existence and uniqueness result. To this end, we define

$$p^* := \frac{2}{1-2\gamma}.$$

It is well known that a contraction in a complete metric space admits a unique fixed point. We need the following lemma concerning the continuity of fixed points for parametrical contractions.

Lemma 3.5. *Consider a Banach space $(Y,|\cdot|_Y)$ and a metric space (U,d), let $0\leq\alpha<1$ and let us consider maps $h(u,\cdot): U\times Y\to Y$, $h_n(u,\cdot): U\times Y\to Y$, where $n\in\mathbb{N}$. Assume that $h(u,\cdot)$ and $h_n(u,\cdot)$ are α-contractions for each $u\in U$ and each $n\in\mathbb{N}$. Given $u\in U$, call $\varphi(u)$ and $\varphi_n(u)$ the unique fixed points of $h(u,\cdot)$ and $h_n(u,\cdot)$, respectively.*

(i) *If $h_n\to h$ pointwise on $U\times Y$, then $\varphi_n\to\varphi$ pointwise on U.*

(ii) *If there exists an increasing concave function w on $\mathbb{R}^+$ such that $w(0)=0$ and*

$$|h(u,y)-h(v,y)|_Y \ \leq \ w(d(u,v)), \qquad \forall\, u,v\in U,\ y\in Y, \tag{3.5}$$

then

$$|\varphi(u)-\varphi(v)|_Y \ \leq \ \frac{1}{1-\alpha}w(d(u,u')), \qquad \forall\, u,v\in U.$$

Proof. From the assumption that $h_n(u,\cdot)$ and $h_n(v,\cdot)$ are α contractions for all $u,v\in U$ and $n\in\mathbb{N}$ we deduce

$$|\varphi_n(u)-\varphi(v)|\leq\frac{|h_n(u,\varphi(u)))-h(v,\varphi(v))|}{1-\alpha},\quad |\varphi(u)-\varphi(v)|\leq\frac{|h(u,\varphi(u))-h(v,\varphi(v))|}{1-\alpha}.$$

The latter yields (i) by taking $u=v$ and letting $n\to\infty$, and (ii) by using also (3.5). □

Theorem 3.6. *Let Assumption 3.1 hold. Then, for every $p>p^*$, $t\in[0,T]$ and $Z\in\mathcal{H}^p_{\mathcal{P}}(H)$, there exists a unique mild solution $X^{t,Z}$ to* (3.1)*. Moreover, $X^{t,Z}\in\mathcal{H}^p_{\mathcal{P}}(H)$ and*

$$\|X^{t,Z}\|_{\mathcal{H}^p_{\mathcal{P}}(H)} \ \ \leq \ \ K_0(1+\|Z\|_{\mathcal{H}^p_{\mathcal{P}}(H)}), \qquad \forall\,(t,Z)\in[0,T]\times\mathcal{H}^p_{\mathcal{P}}(H). \tag{3.6}$$

Finally, the map

$$[0,T]\times\mathcal{H}^p_{\mathcal{P}}(H)\to\mathcal{H}^p_{\mathcal{P}}(H), \quad (t,Z)\mapsto X^{t,Z} \tag{3.7}$$

is Lipschitz continuous with respect to Z, uniformly in $t\in[0,T]$, and jointly continuous.

Remark 3.7. *Since for $p^* < p < q$ we have $\mathcal{H}^p_{\mathcal{P}}(H) \supset \mathcal{H}^q_{\mathcal{P}}(H)$, if $Z \in \mathcal{H}^q_{\mathcal{P}}(H)$, then the associated mild solution $X^{t,Z}$ is also a solution in $\mathcal{H}^p_{\mathcal{P}}(H)$ and, by uniqueness, it is* the *solution in that space. Hence, the solution does not depend on the specific $p > p^*$ chosen.*

Proof of Theorem 3.6. As far as we know, a reference exactly fitting the result above is not available in literature. For brevity, we only sketch the proof, as the arguments are quite standard but rather technical (for the first part of the claim, the closest reference, for the non path-dependent case, is [20, Prop. 3.2]).

Fix $p > p^*$. Let $\beta > 0$ and let us introduce the equivalent norm in $\mathcal{H}^p_{\mathcal{P}}(H)$

$$\|Y\|_{p,\beta} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} e^{-\beta t}\,|Y_t|^p\right]\right)^{1/p}.$$

Let $t \in [0,T]$ and define, for $Z, Y \in \mathcal{H}^p_{\mathcal{P}}(H)$, the process

$$\begin{aligned}[][\Phi_t(Z,Y)]_s \;&:=\; \mathbb{1}_{[0,t)}(s)Z_s + \mathbb{1}_{[t,T]}(s)e^{(s-t)A}Z_t \\ &\quad + \int_t^{t\vee s} e^{(s-r)A}b(r,Y)dr + \int_t^{t\vee s} e^{(s-r)A}\sigma(r,Y)dW_r, \quad s \in [0,T]. \end{aligned} \tag{3.8}$$

As in [20, Prop. 3.2]), using the so called factorization method of [8, Theorem 5.10] and the Burkholder-Davis-Gundy inequality, one shows that $(Z,Y) \mapsto \Phi_t(Z,Y)$ defines a map

$$\Phi_t \;:\; \mathcal{H}^p_{\mathcal{P}}(H) \times \mathcal{H}^p_{\mathcal{P}}(H) \;\longrightarrow\; \mathcal{H}^p_{\mathcal{P}}(H)$$

and that, for suitably large β, there exist constants $\alpha \in [0,1)$ and $C_0 > 0$, both independent of t, such that

$$\|\Phi_t(Z_1,Y_1) - \Phi_t(Z_2,Y_2)\|_{p,\beta} \;\leq\; C_0\|Z_1 - Z_2\|_{p,\beta} + \alpha\|Y_1 - Y_2\|_{p,\beta}. \tag{3.9}$$

This proves that, for each $t \in [0,T]$ and $Z \in \mathcal{H}^p_{\mathcal{P}}(H)$, there exists a unique mild solution $X^{t,Z}$ in the space $\mathcal{H}^p_{\mathcal{P}}(H)$ to (3.1). Moreover, by applying Lemma 3.5(ii), we also get that the map $[0,T]\times\mathcal{H}^p_{\mathcal{P}}(H) \to \mathcal{H}^p_{\mathcal{P}}(H)$, $(t,Z) \mapsto X^{t,Z}$ is Lipschitz continuous with respect to Z, uniformly in $t \in [0,T]$.

To show the continuity of the latter map with respect to $t \in [0,T]$, pick a sequence $t_n \to t$. One can prove that $\Phi_{t_n} \to \Phi_t$ pointwise. Moreover, there exist $\beta > 0$ and $\alpha \in [0,1)$, independent of n, such that the maps Φ_{t_n} are α-contractions with respect to the second variable. Hence, applying Lemma 3.5(i), one gets that $X^{t_n,Z} \to X^{t,Z}$ in $\mathcal{H}^p_{\mathcal{P}}(H)$. Consequently, recalling the proved uniform (in $t \in [0,T]$) Lipschitz continuity of the map $\mathcal{H}^p_{\mathcal{P}}(H) \to \mathcal{H}^p_{\mathcal{P}}(H)$, $Z \mapsto X^{t,Z}$, we conclude that the latter map is jointly continuous. Finally, (3.6) follows from the continuity properties proved above. □

We notice that uniqueness of mild solutions yields the flow property for the solution with initial data $(t,\mathbf{x}) \in \Lambda$:

$$X^{t,\mathbf{x}} = X^{s,X^{t,\mathbf{x}}}, \quad \forall\,(t,\mathbf{x}) \in \Lambda,\ \forall\, s \in [t,T]. \tag{3.10}$$

In the sequel, we shall use the following generalized dominated convergence result.

Lemma 3.8. Let (Σ, μ) be a measure space. Assume that $f_n, g_n, f, g \in L^1(\Sigma, \mu; \mathbb{R})$, $f_n \to f$ and $g_n \to g$ μ-a.e., $|f_n| \leq g_n$ and $\int_\Sigma g_n d\mu \to \int_\Sigma g d\mu$. Then $\int_\Sigma f_n d\mu \to \int_\Sigma f d\mu$.

Corollary 3.9. *Let $p' \geq 1$, $\kappa \in L^\infty(\Omega, \mathcal{F}, \mathbb{P}; C_{p'}(\Lambda))$ and $p > p^*$, $p \geq p'$. Then the map*

$$[0,T] \times [0,T] \times \mathcal{H}^p_{\mathcal{P}}(H) \to \mathbb{R}, \quad (s,t,Z) \mapsto \mathbb{E}\left[\kappa(\cdot)(s, X^{t,Z})\right] \tag{3.11}$$

is well-defined and continuous.

Proof. In view of Theorem 3.6, the map (3.11) is well-defined. Concerning continuity, again in view of Theorem 3.6, it suffices to show that the map

$$[0,T] \times \mathcal{H}^p_{\mathcal{P}}(H) \to \mathbb{R}, \quad (s,Y) \mapsto \mathbb{E}[\kappa(\cdot)(s,Y)]$$

is continuous. Let $\{Y^{(n)}\}_n$ be a sequence converging to Y in $\mathcal{H}^p_{\mathcal{P}}(H)$, and $s_n \to s$ in $[0,T]$. Let $\{Y^{(n_k)}\}_k$ be a subsequence such that $\|Y - Y^{(n_k)}\|_\infty \to 0$ $\mathbb{P}$-a.s.. Then, using the continuity of $\kappa(\omega)(\cdot,\cdot)$ we get, by applying Lemma 3.8, the convergence $\mathbb{E}[\kappa(\cdot)(s_{n_k}, Y^{(n_k)})] \to \mathbb{E}[\kappa(\cdot)(s,Y)]$. Since the original converging sequence $\{(s_n, Y^{(n)})\}_n$ was arbitrary, we get the claim. □

The following stability result for SDE (3.1) will be used to prove the stability of viscosity solutions in the next section.

Proposition 3.10. *Let Assumption 3.1 hold and assume that it holds also, for each $n \in \mathbb{N}$, for analogous objects A_n, b_n and σ_n, such that the estimates of parts (ii)-(iii) in Assumption 3.1 hold with the constants $M, \hat{M}, \gamma$. Assume that the following convergences hold for every $(t, \mathbf{x}) \in \Lambda$ and every $s \in [0,T]$:*

(i) $e^{sA_n}\mathbf{x}_s \to e^{sA}\mathbf{x}_s$ *in H;*

(ii) $e^{sA_n}b_n(t,\mathbf{x}) \to e^{sA}b(t,\mathbf{x})$ *in H;*

(iii) $e^{sA_n}\sigma_n(t,\mathbf{x}) \to e^{sA}\sigma(t,\mathbf{x})$ *in $L_2(K;H)$.*

Let $t \in [0,T]$ and $Z \in \mathcal{H}^p_{\mathcal{P}}(H)$, with $p > p^*$. Then, calling $X^{(n),t,Z}$ the mild solution to (3.1), where A, b, σ are replaced by A_n, b_n, σ_n, one has the convergence $X^{(n),t,Z} \overset{n\to\infty}{\longrightarrow} X^{t,Z}$ in $\mathcal{H}^p_{\mathcal{P}}(H)$ and, for fixed t, there exists K_0 such that

$$\|X^{(n),t,Z}\|_{\mathcal{H}^p_{\mathcal{P}}(H)} \quad \leq \quad K_0(1 + \|Z\|_{\mathcal{H}^p_{\mathcal{P}}(H)}), \qquad \forall\, Z \in \mathcal{H}^p_{\mathcal{P}}(H),\, n \in \mathbb{N}. \tag{3.12}$$

Proof. Let $(t,Z) \in [0,T] \times \mathcal{H}^p_{\mathcal{P}}(H)$. Construct maps $\Phi_{t,n}$ analogous to the map (3.8), but with coefficients A_n, b_n, σ_n. Then there exist $C_0 > 0$ and $\alpha \in [0,1)$ independent of n such that (3.9) holds for all $\Phi_{t,n}$, $n \in \mathbb{N}$. As indicated in the proof of Theorem 3.6, using the factorization method and the Burkholder-Davis-Gundy inequality, one shows, by dominated convergence, the pointwise convergence of $\Phi_{t,n}$ to Φ_t. Applying Lemma 3.5(i), we conclude that $X^{(n),t,Z} \to X^{t,Z}$ in $\mathcal{H}^p_{\mathcal{P}}(H)$. Finally, for fixed t, by the fact that (3.9) holds for all $\Phi_{t,n}$ when $n \in \mathbb{N}$, and by applying Lemma 3.5(ii), we obtain that $X^{(n),t,Z}$ is Lipschitz in Z, uniformly in n. This last continuity, jointly with the fact that $X^{(n),t,\mathbf{0}} \to X^{t,\mathbf{0}}$ in $\mathcal{H}^p_{\mathcal{P}}(H)$, gives (3.12). □

4 Path-dependent PDEs and viscosity solutions in Hilbert spaces

In the present section, we introduce a path-dependent PDE in the space H and study it through the concept of viscosity solutions in the spirit of the definition given in [13, 14, 33]. As in [33], we also provide an equivalent definition in terms of jets. The key result is a martingale characterization for viscosity sub/supersolution, from which follows a stability result and the comparison principle. We finally prove the existence of a viscosity solution through a fixed point argument.

4.1 Definition: test functions and semijets

We begin introducing the set $C_X^{1,2}(\Lambda)$ of smooth functions, which will be used to define test functions. We note that the definition of the latter set shall depend on the process $X^{t,\mathbf{x}}$ solution to (3.1), that is on the coefficients A, b, σ. The subscript X in the notation $C_X^{1,2}(\Lambda)$ stays there to recall that.

Definition 4.1. *We say that $u \in C_X^{1,2}(\Lambda)$ if there exists $p \geq 1$ such that $u \in C_p(\Lambda)$ and there exist $\alpha \in C_p(\Lambda)$, $\beta \in C_p(\Lambda; H)$ such that*

$$du(s, X^{t,\mathbf{x}}) \;=\; \alpha(s, X^{t,\mathbf{x}})ds + \langle \beta(s, X^{t,\mathbf{x}}), dW_s \rangle, \quad \forall (t, \mathbf{x}) \in \Lambda,\ \forall s \in [t, T]. \tag{4.1}$$

Notice that α and β in Definition 4.1 are uniquely determined, as it can be easily shown by identifying the finite variation part and the Brownian part in (4.1). Given $u \in C_X^{1,2}(\Lambda)$, we denote

$$\mathcal{L}u \;:=\; \alpha.$$

We refer to Appendix A for an insight on the above notation for α and for a link with the pathwise derivatives introduced in [12].

Remark 4.2. *One of the key ingredients of the notion of viscosity solution we are going to define is the concept of test function introduced in Definition 4.1. Notice that, the larger the class of test functions, the easier should be the proof of the comparison principle and the harder the proof of the existence. In order to make easier the proof of uniqueness, we weaken the concept of test functions as much as possible – but, clearly, still keeping "safe" the existence part. The space $C_X^{1,2}(\Lambda)$ is the result of this trade-off. It is a quite large class of test functions: for example, as it will be shown in Lemma 4.12 below, if $f \in C_p(\Lambda)$, $p \geq 1$, then $\varphi(t, \mathbf{x}) := \int_0^t f(s, \mathbf{x})ds$ is in $C_X^{1,2}(\Lambda)$, whereas, even if $H = \mathbb{R}^n$ and f is Markovian (i.e., $f(s, \mathbf{x}) = f(s, \mathbf{x}_s)$), it does not belong, in general, to the usual class $C^{1,2}(\mathbb{R}^n; \mathbb{R})$ of smooth functions.* □

We are concerned with the study the following *path-dependent* PDE (from now on, PPDE):

$$\mathcal{L}u(t, \mathbf{x}) + F(t, \mathbf{x}, u(t, \mathbf{x})) \;=\; 0, \qquad (t, \mathbf{x}) \in \Lambda,\ t < T, \tag{4.2}$$

with terminal condition

$$u(T, \mathbf{x}) \;=\; \xi(\mathbf{x}), \quad \mathbf{x} \in \mathbb{W}, \tag{4.3}$$

where $F\colon \Lambda \times \mathbb{R} \to \mathbb{R}$ and $\xi\colon \mathbb{W} \to \mathbb{R}$.

Let us introduce the concept of viscosity solution for the path-dependent PDE (4.2), following [13, 14, 33]. To this end, we denote

$$\mathcal{T} \;:=\; \big\{\tau\colon \Omega \to [0,T] \mid \tau \text{ is an } \mathbb{F}\text{-stopping time}\big\}.$$

Given $u \in C_p(\Lambda)$ for some $p \geq 1$, we define the following two classes of test functions:

$$\underline{\mathcal{A}}u(t,\mathbf{x}) := \Big\{\varphi \in C^{1,2}_X(\Lambda)\colon \text{there exists } \mathrm{H} \in \mathcal{T},\ \mathrm{H} > t,\ \text{such that} \\ (\varphi - u)(t,\mathbf{x}) = \min_{\tau \in \mathcal{T},\, \tau \geq t} \mathbb{E}\big[(\varphi - u)(\tau \wedge \mathrm{H}, X^{t,\mathbf{x}})\big]\Big\},$$

$$\overline{\mathcal{A}}u(t,\mathbf{x}) := \Big\{\varphi \in C^{1,2}_X(\Lambda)\colon \text{there exists } \mathrm{H} \in \mathcal{T},\ \mathrm{H} > t,\ \text{such that} \\ (\varphi - u)(t,\mathbf{x}) = \max_{\tau \in \mathcal{T},\, \tau \geq t} \mathbb{E}\big[(\varphi - u)(\tau \wedge \mathrm{H}, X^{t,\mathbf{x}})\big]\Big\}.$$

Definition 4.3. *Let $u \in C_p(\Lambda)$ for some $p \geq 1$.*

i) We say that u is a ***viscosity subsolution*** *(resp.* ***supersolution****) of the path-dependent PDE (4.2) if*

$$-\mathcal{L}\varphi(t,\mathbf{x}) - F(t,\mathbf{x},u(t,\mathbf{x})) \;\leq\; 0, \quad (\textit{resp. } \geq 0)$$

for any $(t,\mathbf{x}) \in \Lambda$, $t < T$, and any $\varphi \in \underline{\mathcal{A}}u(t,\mathbf{x})$ (resp. $\varphi \in \overline{\mathcal{A}}u(t,\mathbf{x})$).

ii) We say that u is a ***viscosity solution*** *of the path-dependent PDE (4.2) if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 4.4. *As usual, in Definition 4.3, without loss of generality, one can consider only the test functions $\varphi \in \underline{\mathcal{A}}u(t,\mathbf{x})$ (resp. $\overline{\mathcal{A}}u(t,\mathbf{x})$) such that $(\varphi - u)(t,\mathbf{x}) = 0$.* □

Remark 4.5. *The notion of viscosity solution we introduced is designed for our path-dependent PDE and it should be modified in a suitable way if we want to consider more general nonlinearities. For example, if we take F depending also on $\partial_{\mathbf{x}}u$ as in [13], this would entail a substantial change in our definition of viscosity solution. In [13] this corresponds to take an optimal stopping problem under nonlinear expectation, i.e., under a family of probability measures; in our formalism which separates the (fixed) probability space from the state space (see Remark 2.5), this would correspond to take a mixed control/stopping problem, with the control acting on the drift of the SDE. In our infinite-dimensional framework, the case under study already presents some specific difficulties and interesting features – for instance, already in the comparison with the literature on viscosity solutions in infinite dimension in the Markovian case, see Section 5 – so we leave the investigations of these generalizations for future research.* □

Following [33], we now provide an equivalent definition of viscosity solution in terms of semijets. Given $u \in C_p(\Lambda)$, for some $p \geq 1$, define the *subjet* and *superjet* of u at $(t, \mathbf{x}) \in \Lambda$ as

$$\begin{aligned}
\underline{\mathcal{J}}u(t,\mathbf{x}) &:= \big\{\alpha \in \mathbb{R} \colon \exists\, \varphi \in \underline{\mathcal{A}}u(t,\mathbf{x}) \text{ such that } \varphi(s,\mathbf{y}) = \alpha s,\ \forall\,(s,\mathbf{y}) \in \Lambda\big\},\\
\overline{\mathcal{J}}u(t,\mathbf{x}) &:= \big\{\alpha \in \mathbb{R} \colon \exists\, \varphi \in \overline{\mathcal{A}}u(t,\mathbf{x}) \text{ such that } \varphi(s,\mathbf{y}) = \alpha s,\ \forall\,(s,\mathbf{y}) \in \Lambda\big\}.
\end{aligned}$$

We have the following equivalence result.

Proposition 4.6. *Suppose that Assumption 3.1 holds. Then $u \in C_p(\Lambda)$, $p \geq 1$, is a viscosity subsolution (resp. supersolution) of the path-dependent PDE (4.2) if and only if:*

$$-\alpha - F(t,\mathbf{x},u(t,\mathbf{x})) \ \leq\ 0, \quad (\textit{resp. } \geq 0),$$

for every $\alpha \in \underline{\mathcal{J}}u(t,\mathbf{x})$ (resp. $\alpha \in \overline{\mathcal{J}}u(t,\mathbf{x})$).

Proof. We focus on the *'if'* part, since the other implication is clear. Fix $(t,\mathbf{x}) \in \Lambda$ and $\varphi \in \underline{\mathcal{A}}u(t,\mathbf{x})$ (the supersolution part has a similar proof). From Definition 4.1 we know that there exists $\mathcal{L}\varphi := \alpha \in C_p(\Lambda)$ and $\beta \in C_p(\Lambda; H)$ such that (4.1) holds, with φ in place of u. Set

$$\alpha_0 \ :=\ \mathcal{L}\varphi(t,\mathbf{x}) \ =\ \alpha(t,\mathbf{x})$$

and, for every $\varepsilon > 0$, consider $\varphi_\varepsilon(s,\mathbf{y}) := (\alpha_0 + \varepsilon)s$, for all $(s,\mathbf{y}) \in \Lambda$. Then $\varphi_\varepsilon \in C_X^{1,2}(\Lambda)$. Since $\mathcal{L}\varphi$ is continuous, we can find $\delta > 0$ such that

$$\big|\mathcal{L}\varphi(t',\mathbf{x}') - \alpha_0\big| \ =\ \big|\mathcal{L}\varphi(t',\mathbf{x}') - \mathcal{L}\varphi(t,\mathbf{x})\big| \ \leq\ \varepsilon, \qquad \text{if } \mathbf{d}_\infty\big((t',\mathbf{x}'),(t,\mathbf{x})\big) \ \leq\ \delta.$$

Let H be the stopping time associated to φ appearing in the definition of $\underline{\mathcal{A}}u(t,\mathbf{x})$ and define

$$\mathrm{H}_\varepsilon \ :=\ \mathrm{H} \wedge \big\{s \geq t \colon \mathbf{d}_\infty\big((s,X^{t,\mathbf{x}}),(t,\mathbf{x})\big) \ >\ \delta\big\}.$$

Note that $\mathrm{H}_\varepsilon > 0$. Then, for any $\tau \in \mathcal{T}$ with $\tau \geq t$, we have

$$\begin{aligned}
&(u-\varphi_\varepsilon)(t,\mathbf{x}) - \mathbb{E}\big[(u-\varphi_\varepsilon)(\tau\wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\big]\\
&= (u-\varphi)(t,\mathbf{x}) - \mathbb{E}\big[(u-\varphi)(\tau\wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\big] + \mathbb{E}\big[(\varphi_\varepsilon-\varphi)(\tau\wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\big] - (\varphi_\varepsilon-\varphi)(t,\mathbf{x})\\
&\geq \mathbb{E}\big[(\varphi_\varepsilon-\varphi)(\tau\wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\big] - (\varphi_\varepsilon-\varphi)(t,\mathbf{x}),
\end{aligned} \tag{4.4}$$

where the last inequality follows from the fact that $\varphi \in \underline{\mathcal{A}}u(t,\mathbf{x})$. Since φ and φ_ε belong to $C_X^{1,2}(\Lambda)$, we can write

$$\mathbb{E}\left[\varphi(\tau\wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\right] \ =\ \varphi(t,\mathbf{x}) + \mathbb{E}\left[\int_t^{\tau\wedge \mathrm{H}_\varepsilon} \mathcal{L}\varphi(s,X^{t,\mathbf{x}})ds\right] \tag{4.5}$$

and, clearly, we also have

$$\mathbb{E}\left[\varphi_\varepsilon(\tau\wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\right] =\ \varphi_\varepsilon(t,\mathbf{x}) + \mathbb{E}\left[\int_t^{\tau\wedge \mathrm{H}_\varepsilon} (\alpha_0+\varepsilon)ds\right]. \tag{4.6}$$

Plugging (4.5) and (4.6) into (4.4), we obtain

$$(\varphi_\varepsilon - u)(t,\mathbf{x}) - \mathbb{E}\big[(\varphi_\varepsilon - u)(\tau\wedge \mathrm{H}_\varepsilon, X^{t,\mathbf{x}})\big] \;\leq\; \mathbb{E}\left[\int_t^{\tau\wedge\mathrm{H}_\varepsilon} \big(\mathcal{L}\varphi(s,X^{t,\mathbf{x}}) - (\alpha_0+\varepsilon)\big)ds\right] \;\leq\; 0,$$

where the last inequality follows by definition of H_ε. It follows that $\varphi_\varepsilon \in \underline{\mathcal{A}}(t,\mathbf{x})$, hence that $\alpha_0+\varepsilon \in \underline{\mathcal{J}}u(t,\mathbf{x})$, therefore

$$-(\mathcal{L}\varphi(t,\mathbf{x})+\varepsilon) - F(t,\mathbf{x},u(t,\mathbf{x})) \;=\; -(\alpha_0+\varepsilon) - F(t,\mathbf{x},u(t,\mathbf{x})) \;\leq\; 0.$$

By arbitrariness of ε we conclude. □

4.2 Martingale characterization and stability

In the sequel, we shall consider the following conditions on F.

Assumption 4.7.

(i) $F\colon \Lambda\times\mathbb{R}\to\mathbb{R}$ *is continuous and satisfies the growth condition: there exists* $L>0$ *such that*

$$|F(t,\mathbf{x},y)| \leq L(1+\|\mathbf{x}\|_\infty^p + |y|), \qquad \forall\,(t,\mathbf{x})\in\Lambda,\;\forall\,y\in\mathbb{R}. \tag{4.7}$$

(ii) F *is Lipschitz with respect to the third variable, uniformly in the other ones: there exists* $\hat{L}>0$ *such that*

$$|F(t,\mathbf{x},y) - F(t,\mathbf{x},y')| \leq \hat{L}|y-y'|, \qquad \forall\,(t,\mathbf{x})\in\Lambda,\;\forall\,y,y'\in\mathbb{R}. \tag{4.8}$$

We now state the main result of this section, the sub(super)martingale characterization for viscosity sub(super)solutions of PPDE (4.2).

Theorem 4.8. *Let* $u\in C_p(\Lambda)$, $p\geq 1$, *and let Assumptions 3.1 and 4.7(i) hold. The following facts are equivalent.*

(i) *For every* $(t,\mathbf{x})\in\Lambda$

$$u(t,\mathbf{x}) \;\leq\; \mathbb{E}\bigg[u(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\bigg], \quad \forall\,s\in[t,T], \tag{4.9}$$

(resp., $\geq$*).*

(ii) *For every* $(t,\mathbf{x})\in\Lambda$ *with* $t<T$ *the process*

$$\bigg(u(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\bigg)_{s\in[t,T]} \tag{4.10}$$

is a $(\mathcal{F}_s)_{s\in[t,T]}$*-submartingale (resp., supermartingale).*

(iii) u *is a viscosity subsolution (resp., supersolution) of PPDE (4.2).*

To prove Theorem 4.8 we need some technical results from the optimal stopping theory. For this reason we look at them. Let $u, f \in C_p(\Lambda)$ for some $p \geq 1$. Given $s \in [0,T]$, define $\Lambda_s := \{(t,\mathbf{x}) \in \Lambda \mid t \in [0,s]\}$ and consider the optimal stopping problems

$$\Psi_s(t,\mathbf{x}) := \sup_{\tau \in \mathcal{T},\, \tau \geq t} \mathbb{E}\left[u(\tau \wedge s, X^{t,\mathbf{x}}) + \int_t^{\tau \wedge s} f(r, X^{t,\mathbf{x}})dr\right], \qquad (t,\mathbf{x}) \in \Lambda_s. \tag{4.11}$$

Remark 4.9. *Notice that in the present paper we need only to consider optimal stopping problems (4.11) with deterministic finite horizon $s \in [0,T]$, rather than random finite horizon as in [33].* □

Lemma 4.10. *Let Assumption 3.1 hold and let $u, f \in C_p(\Lambda)$ for some $p \geq 1$. Then Ψ_s is lower semicontinuous on Λ_s.*

Proof. Using the fact that $u, f \in C_p(\Lambda)$ for some $p \geq 1$, we see, by Corollary 3.9, that the functional

$$\Lambda_s \to \mathbb{R}, \qquad (t,\mathbf{x}) \mapsto \mathbb{E}\left[u((\tau \wedge s) \vee t, X^{t,\mathbf{x}}) + \int_t^{(\tau \wedge s)\vee t} f(r, X^{t,\mathbf{x}})dr\right]$$

is well-defined and continuous for every $\tau \in \mathcal{T}$. We deduce that

$$\begin{aligned} \Psi_s(t,\mathbf{x}) &= \sup_{\tau \in \mathcal{T},\, \tau \geq t} \mathbb{E}\left[u(\tau \wedge s, X^{t,\mathbf{x}}) + \int_t^{\tau \wedge s} f(r, X^{t,\mathbf{x}})dr\right] \\ &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[u((\tau \wedge s) \vee t, X^{t,\mathbf{x}}) + \int_t^{(\tau \wedge s)\vee t} f(r, X^{t,\mathbf{x}})dr\right], \qquad (t,\mathbf{x}) \in \Lambda_s, \end{aligned} \tag{4.12}$$

is lower semicontinuous, as it is supremum of continuous functions. □

Define the continuation region

$$\mathcal{C}_s := \{(t,\mathbf{x}) \in \Lambda_s \mid \Psi_s(t,\mathbf{x}) > u(t,\mathbf{x})\}.$$

Due to the continuity of u and the lower semicontinuity of Ψ_s, it follows that $\mathcal{C}_s$ is an open subset of Λ_s. From the general theory of optimal stopping we have the following result.

Theorem 4.11. *Let Assumption 3.1 hold. Let $s \in [0,T]$, $(t,\mathbf{x}) \in \Lambda_s$ and define the random time $\tau^*_{t,\mathbf{x}} := \inf\big\{r \in [t,s] \colon (r, X^{t,\mathbf{x}}) \notin \mathcal{C}_s\big\}$, with the convention $\inf \emptyset = s$. Then $\tau^*_{t,\mathbf{x}}$ is the first optimal stopping time for problem (4.11).*

Proof. First of all, we notice that, since $u, f \in C_p(\Lambda)$ for some $p \geq 1$, by (O5) we have, for every $(t,\mathbf{x}) \in \Lambda$,

$$\mathbb{E}\left[\sup_{r \in [t,T]} |u(r, X^{t,\mathbf{x}})|\right] < +\infty, \qquad \mathbb{E}\left[\int_t^T |f(r, X^{t,\mathbf{x}})|dr\right] < +\infty. \tag{4.13}$$

Now, given $(t,\mathbf{x}) \in \Lambda$, consider the *window process*

$$[0,T] \times \Omega \longrightarrow \mathbb{W}, \quad (r,\omega) \longmapsto \mathbb{X}_r^{t,\mathbf{x}}(\omega),$$

where

$$\mathbb{X}_r^{t,\mathbf{x}}(\omega)(\alpha) := \begin{cases} \mathbf{x}_0, & \alpha + r < 0, \\ X_{\alpha+r}^{t,\mathbf{x}}(\omega), & \alpha + r \geq 0, \end{cases} \qquad r \in [0,T], \ \alpha \in [-T,0].$$

Clearly this process is Markovian and we can write the optimal stopping problem in terms of it. Then, the standard theory of optimal stopping of Markovian processes allows to conclude. More precisely, taking into account (4.13), we can use Corollary 2.9, Ch. I.1, of [32] when $f = 0$; when $f \neq 0$, the integral part of the functional can be reduced to u by adding one dimension to the problem in a standard way (see, e.g., Ch. III.6 in [32]) $\square$

Lemma 4.12. *Let Assumption 3.1 hold. Let $u, f \in C_p(\Lambda)$ and assume that there exist $s \in [0,T]$ and $(t,\mathbf{x}) \in \Lambda_s$, with $t < s$, such that*

$$u(t,\mathbf{x}) \ > \ \mathbb{E}\left[u(s, X^{t,\mathbf{x}}) + \int_t^s f(r, X^{t,\mathbf{x}})dr\right] \qquad (\textit{resp.} \ <). \tag{4.14}$$

Then there exists $(a,\mathbf{y}) \in \Lambda_s$ such that the function φ defined as $\varphi(s,\mathbf{z}) := -\int_0^s f(r,\mathbf{z})dr$ belongs to $\underline{\mathcal{A}}u(a,\mathbf{y})$ (resp. belongs to $\overline{\mathcal{A}}u(a,\mathbf{y})$).

Proof. We prove the claim for the "sub-part". The proof of the "super-part" is completely symmetric.

First, we notice that $\varphi \in C_X^{1,2}(\Lambda)$, as it satisfies (4.1) with $\alpha = -f$ and $\beta \equiv 0$. Let us now focus on the maximum property. Consider the optimal stopping problem (4.11) and let $\tau_{t,\mathbf{x}}^*$ be the stopping time of Theorem 4.11. Due to (4.14) we have $\mathbb{P}\{\tau_{t,\mathbf{x}}^* < s\} > 0$. This implies that there exists $(a,\mathbf{y}) \in \Lambda_s \setminus \mathcal{C}_s$. Hence

$$-u(a,\mathbf{y}) = -\Psi_s(a,\mathbf{y}) = \min_{\tau \in \mathcal{T}, \tau \geq a} \mathbb{E}\left[-\int_a^{\tau \wedge s} f(r, X^{a,\mathbf{y}})dr - u(\tau \wedge s, X^{a,\mathbf{y}})\right].$$

By adding $-\int_0^a f(r,\mathbf{y})dr$ to the above equality, we get the claim ([5]). $\square$

Proof of Theorem 4.8. We prove the claim for the case of the subsolution/submartingale. The other claim can be proved in a completely symmetric way.

$(i) \Rightarrow (ii)$. We need to prove that, for every pair of times (s_1, s_2) with $t \leq s_1 \leq s_2 \leq T$,

$$u(s_1, X^{t,\mathbf{x}}) \ \leq \ \mathbb{E}\left[u(s_2, X^{t,\mathbf{x}}) + \int_{s_1}^{s_2} F(r, X^{t,\mathbf{x}}, u(r, X^{t,\mathbf{x}}))dr \ \Big| \ \mathcal{F}_{s_1}\right]. \tag{4.15}$$

[5] The role of the localizing stopping time $\textsc{h}$ in the definition of test functions is here played by s.

Using (3.10) and the equality $X^{s_1,X^{t,\mathbf{x}}} = X^{s_1,X^{t,\mathbf{x}}_{\cdot\wedge s_1}}$, we have ([6])

$$\mathbb{E}\left[u(s_2,X^{t,\mathbf{x}}) + \int_{s_1}^{s_2} F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr \;\Big|\; \mathcal{F}_{s_1}\right]$$
$$= \mathbb{E}\Bigg[u(s_2,X^{s_1,X^{t,\mathbf{x}}_{\cdot\wedge s_1}}) + \int_{s_1}^{s_2} F(r,X^{s_1,X^{t,\mathbf{x}}_{\cdot\wedge s_1}},u(r,X^{s_1,X^{t,\mathbf{x}}_{\cdot\wedge s_1}}))dr \;\Big|\; \mathcal{F}_{s_1}\Bigg].$$

Note that $X^{s_1,\mathbf{x}'}$ is independent of $\mathcal{F}_{s_1}$ for each $\mathbf{x}'$ and $X^{t,\mathbf{x}}_{\cdot\wedge s_1}$ is $\mathcal{F}_{s_1}$-measurable. Hence, using [1, Lemma 3.9, p. 55],

$$\mathbb{E}\Bigg[u(s_2,X^{s_1,X^{t,\mathbf{x}}_{\cdot\wedge s_1}}) + \int_{s_1}^{s_2} F(r,X^{s_1,X^{t,\mathbf{x}}_{\cdot\wedge s_1}},u(r,X^{s_1,X^{t,\mathbf{x}}_{\cdot\wedge s_1}}))dr \;\Big|\; \mathcal{F}_{s_1}\Bigg]$$
$$= \mathbb{E}\Bigg[u(s_2,X^{s_1,\mathbf{x}'}) + \int_{s_1}^{s_2} F(r,X^{s_1,\mathbf{x}'},u(r,X^{s_1,\mathbf{x}'}))dr\Bigg]\Bigg|_{\mathbf{x}'=X^{t,\mathbf{x}}}$$

Now we conclude, as (i) holds.

(ii) $\Rightarrow$ *(iii)*. Let $\varphi \in \underline{\mathcal{A}}(t,\mathbf{x})$. Then, by definition of test function, there exists $\mathrm{H} \in \mathcal{T}$, with $\mathrm{H} > t$, such that

$$(\varphi - u)(t,\mathbf{x}) \;\geq\; \mathbb{E}\left[(\varphi - u)\left(\tau\wedge\mathrm{H},X^{t,\mathbf{x}}\right)\right], \qquad \forall\tau\in\mathcal{T},\, t\leq\tau. \tag{4.16}$$

As $\varphi \in C^{1,2}_X(\Lambda)$, we can write

$$\mathbb{E}\left[\varphi(\tau\wedge\mathrm{H},X^{t,\mathbf{x}})\right] \;=\; \varphi(t,\mathbf{x}) + \mathbb{E}\left[\int_t^{\tau\wedge\mathrm{H}} \mathcal{L}\varphi(s,X^{t,\mathbf{x}})ds\right] \tag{4.17}$$

Combining (4.16)-(4.17), we get

$$-\mathbb{E}\left[\int_t^{\tau\wedge\mathrm{H}} \mathcal{L}\varphi(s,X^{t,\mathbf{x}})ds\right] \;\leq\; u(t,\mathbf{x}) - \mathbb{E}\left[u(\tau\wedge\mathrm{H},X^{t,\mathbf{x}})\right]$$

or, equivalently,

$$-\,\mathbb{E}\left[\int_t^{\tau\wedge\mathrm{H}} \left(\mathcal{L}\varphi(s,X^{t,\mathbf{x}}) + F(s,X^{t,\mathbf{x}},u(s,X^{t,\mathbf{x}}))\right)ds\right]$$
$$\leq\; u(t,\mathbf{x}) - \mathbb{E}\left[u(\tau\wedge\mathrm{H},X^{t,\mathbf{x}}) + \int_t^{\tau\wedge\mathrm{H}} F(s,X^{t,\mathbf{x}},u(s,X^{t,\mathbf{x}}))ds\right]. \tag{4.18}$$

Now observe that the submartingale assumption (4.10) implies that the right-hand side of (4.18) is smaller than 0. Hence, we can conclude by considering in (4.18) stopping times of the form $\tau = t+\varepsilon$, with $\varepsilon > 0$, dividing by ε and letting $\varepsilon \to 0^+$.

(iii) $\Rightarrow$ *(i)*. Let $\varepsilon > 0$ and consider the function $u_\varepsilon(r,\mathbf{z}) := u(r,\mathbf{z}) + \varepsilon r$. Assume that there exist $\varepsilon > 0$, $(t,\mathbf{x}) \in \Lambda$ and $t < s \leq T$ such that

$$u_\varepsilon(t,\mathbf{x}) \;>\; \mathbb{E}\left[u_\varepsilon(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\right]. \tag{4.19}$$

[6] The flow property of $X^{t,\mathbf{x}}$ used here plays the role of the method based on regular conditional probability used in [13, 14, 15].

By applying Lemma 4.12, we get that φ^ε defined as $\varphi^\varepsilon(r,\mathbf{z}) := \varphi(r,\mathbf{z}) - \varepsilon r$, where φ is defined as in Lemma 4.12 taking $f(r,\cdot) := F(r,\cdot,u(r,\cdot))$, belongs to $\underline{\mathcal{A}}u(a,\mathbf{y})$ for some $(a,\mathbf{y})$. By the viscosity subsolution property of u, we then obtain the contradiction $\varepsilon \leq 0$. Hence we deduce that

$$u_\varepsilon(t,\mathbf{x}) \ \leq \ \mathbb{E}\left[u_\varepsilon(s,X^{t,\mathbf{x}})) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\right]. \tag{4.20}$$

As ε is arbitrary in the argument above, we can take $\varepsilon \downarrow 0$ in (4.20), getting (4.9). $\square$

As a direct consequence of the martingale characterization in Theorem 4.8, we have the following stability result.

Proposition 4.13. *Let the assumptions of Proposition 3.10 hold. Let Assumption 4.7(i) hold and assume that it also holds, for each $n \in \mathbb{N}$, for analogous objects F_n with the same constant L. Let $\{u_n, n \in \mathbb{N}\}$ be a bounded subset of $C_p(\Lambda)$ for some $p \geq 1$ and let $u \in C_p(\Lambda)$. Assume that the following convergences hold:*

(i) $F_n(s,\cdot,y) \to F(s,\cdot,y)$ uniformly on compact subsets of $\mathbb{W}$ for each $(s,y) \in [0,T] \times \mathbb{R}$.

(ii) $u_n(s,\cdot) \to u(s,\cdot)$ uniformly on compact subsets of $\mathbb{W}$ for each $s \in [0,T]$.

Finally, assume that, for each $n \in \mathbb{N}$, the function u_n is viscosity subsolution (resp., supersolution) to PPDE (4.2) associated to the coefficients A_n, b_n, σ_n, F_n. Then u is a viscosity subsolution (resp., supersolution) to (4.2) associated to the coefficients A, b, σ, F.

Proof. For any $n > 0$ and $(t,\mathbf{x}) \in \Lambda$, it follows from Proposition 3.6 that there exists a unique mild solution $X^{(n),t,\mathbf{x}}$ to SDE (3.1) with coefficients A_n, b_n, σ_n. By Proposition 3.10

$$X^{(n),t,\mathbf{x}} \overset{n\to\infty}{\longrightarrow} X^{t,\mathbf{x}} \ \text{ in } \ \mathcal{H}^p_{\mathcal{P}}(H), \qquad \forall\,(t,\mathbf{x}) \in \Lambda. \tag{4.21}$$

Since u_n is a viscosity subsolution (the supersolution case can be proved in a similar way) to PPDE (4.2), from statement (i) of Theorem 4.8 we have, for every $(t,\mathbf{x}) \in \Lambda$ with $t < T$,

$$u_n(t,\mathbf{x}) \ \leq \ \mathbb{E}\bigg[u_n(s,X^{(n),t,\mathbf{x}}) + \int_t^s F_n(r,X^{(n),t,\mathbf{x}},u_n(r,X^{(n),t,\mathbf{x}}))dr\bigg], \qquad \forall\, s \in [t,T]. \tag{4.22}$$

In view of the same theorem, to conclude the proof we just need to prove, letting $n \to \infty$, that the same inequality holds true when u_n, F_n and $X^{(n),t,\mathbf{x}}$ are replaced by u, F and $X^{t,\mathbf{x}}$, respectively.

Clearly the left-hand side of the above inequality tends to $u(t,\mathbf{x})$ as $n \to \infty$. Let us consider the right-hand side. From (4.21), up to extracting a subsequence, we have for $\mathbb{P}$-a.e. ω, the convergence $X^{(n),t,\mathbf{x}}(\omega) \to X^{t,\mathbf{x}}(\omega)$ in $\mathbb{W}$. Fix such an ω. Then

$$\mathcal{S}(\omega) := \Big\{X^{(n),t,\mathbf{x}}(\omega)\Big\}_{n\in\mathbb{N}} \ \cup \ \big\{X^{t,\mathbf{x}}(\omega)\big\}$$

is a compact subset of $\mathbb{W}$. Then, for each $s \in [t,T]$,

$$|u_n(s, X^{(n),t,\mathbf{x}}(\omega)) - u(s, X^{t,\mathbf{x}}(\omega))| \leq \sup_{\mathbf{z}\in\mathcal{S}(\omega)} |u_n(s,\mathbf{z}) - u(s,\mathbf{z})| \\ + |u(s, X^{(n),t,\mathbf{x}}(\omega)) - u(s, X^{t,\mathbf{x}}(\omega))| \overset{n\to\infty}{\longrightarrow} 0$$

because $u_n(s,\cdot) \to u(s,\cdot)$ on compact subsets of $\mathbb{W}$, u is continuous and $X^{(n),t,\mathbf{x}}(\omega) \to X^{t,\mathbf{x}}(\omega)$ in $\mathbb{W}$. This shows that $u_n(s, X^{(n),t,\mathbf{x}}(\omega)) \to u(s, X^{t,\mathbf{x}}(\omega))$ for every $s \in [t,T]$. Arguing analogously, we have for each $s \in [t,T]$

$$F_n(s, X^{(n),t,\mathbf{x}}(\omega), u_n(s, X^{(n),t,\mathbf{x}}(\omega))) \overset{n\to\infty}{\longrightarrow} F(s, X^{t,\mathbf{x}}(\omega), u(s, X^{t,\mathbf{x}}(\omega))).$$

Now we can conclude by applying Lemma 3.8. Indeed, assuming without loss of generality $t < s$, the hypotheses are verified for $(\Sigma,\mu) = (\Omega \times [t,s], \mathbb{P} \otimes Leb)$, and

$$\begin{aligned}
f_n(\omega,r) =& \frac{1}{s-t} u_n(s, X^{(n),t,\mathbf{x}}(\omega)) + F_n(r, X^{(n),t,\mathbf{x}}(\omega), u_n(r, X^{(n),t,\mathbf{x}}(\omega))),\\
f(\omega,r) =& \frac{1}{s-t} u(s, X^{t,\mathbf{x}}(\omega)) + F(r, X^{t,\mathbf{x}}(\omega), u(r, X^{t,\mathbf{x}}(\omega))),\\
g_n(\omega,r) =& g_n(\omega) = M'(1 + \|X^{(n),t,\mathbf{x}}(\omega)\|_\infty^p),\\
g(\omega,r) =& g(\omega) = M'(1 + \|X^{t,\mathbf{x}}(\omega)\|_\infty^p),
\end{aligned}$$

for a sufficiently large $M' > 0$, since $\int_\Sigma g_n d\mu \to \int_\Sigma g d\mu$ by (4.21). $\square$

4.3 Comparison principle

In this section we provide a comparison result for viscosity sub and supersolutions of (4.2), which, through the use of a technical lemma provided here, turns out to be a corollary of the characterization of Theorem 4.8.

Lemma 4.14. *Let $Z \in \mathcal{H}^1_{\mathcal{P}}$ and $g\colon [0,T] \times \Omega \times \mathbb{R} \to \mathbb{R}$ be such that $g(\cdot,\cdot,z) \in L^1_{\mathcal{P}}$, for all $z \in \mathbb{R}$, and, for some constant $C_g > 0$,*

$$g(\cdot,\cdot,z) \;\leq\; C_g|z|, \quad \forall\, z \in \mathbb{R}. \tag{4.23}$$

Assume that the process

$$\left(Z_s + \int_t^s g(r,\cdot,Z_r)dr \right)_{s\in[t,T]} \tag{4.24}$$

is an $(\mathcal{F}_s)_{s\in[t,T]}$-submartingale. Then $Z_T \leq 0$, $\mathbb{P}$-a.s., implies $Z_t \leq 0$, $\mathbb{P}$-a.s..

Proof. Let $Z_T \leq 0$ and define

$$\tau^* \;:=\; \inf\{s \geq t : Z_s \leq 0\}.$$

Clearly $t \leq \tau^* \leq T$ and, since Z has continuous trajectories,

$$Z_{\tau^*} \;\leq\; 0. \tag{4.25}$$

Using the submartingale property, we obtain

$$Z_s \;\le\; \mathbb{E}\left[Z_{\tau^*\vee s} + \int_s^{\tau^*\vee s} g(r,\cdot,Z_r)dr \;\middle|\; \mathcal{F}_s \right], \quad \forall\, s \in [t,T]. \tag{4.26}$$

Multiplying (4.26) by the $\mathcal{F}_s$ -measurable random variable $\mathbf{1}_{\{s\le\tau^*\}}$, and recalling (4.25), we find

$$\begin{aligned} \mathbf{1}_{\{s\le\tau^*\}} Z_s \;&\le\; \mathbb{E}\left[\mathbf{1}_{\{s\le\tau^*\}} \left(Z_{\tau^*} + \int_s^{\tau^*} g(r,\cdot,Z_r)dr \right) \;\middle|\; \mathcal{F}_s \right] \\ &\le\; \mathbb{E}\left[\mathbf{1}_{\{s\le\tau^*\}} \int_s^{\tau^*} g(r,\cdot,Z_r)dr \;\middle|\; \mathcal{F}_s \right] \\ &=\; \mathbb{E}\left[\int_s^{T} \mathbf{1}_{\{r\le\tau^*\}} g(r,\cdot,Z_r)dr \;\middle|\; \mathcal{F}_s \right], \quad \forall s \in [t,T]. \end{aligned} \tag{4.27}$$

Now from (4.23) and the definition of τ^*, we have

$$\mathbf{1}_{\{r\le\tau^*\}} g(r,\cdot,Z_r) \;\le\; \mathbf{1}_{\{r\le\tau^*\}} C_g |Z_r| \;=\; \mathbf{1}_{\{r\le\tau^*\}} C_g Z_r, \quad \forall\, r \in [t,T].$$

Plugging the latter inequality into (4.27) and taking the conditional expectations with respect to $\mathcal{F}_t$, we obtain

$$\mathbb{E}\left[\mathbf{1}_{\{s\le\tau^*\}} Z_s \,|\, \mathcal{F}_t \right] \;\le\; C_g \int_s^T \mathbb{E}[\mathbf{1}_{\{r\le\tau^*\}} Z_r \,|\, \mathcal{F}_t] dr, \quad \forall\, s \in [t,T]. \tag{4.28}$$

Now, setting $h(s) := \mathbb{E}[\mathbf{1}_{\{s\le\tau^*\}} Z_s \,|\, \mathcal{F}_t]$, (4.28) becomes

$$h(s) \;\le\; C_g \int_s^T h(r)dr, \quad \forall\, s \in [t,T]. \tag{4.29}$$

Gronwall's Lemma yields $h(s) \le 0$, for all $s \in [t,T]$. In particular, for $s = T$, we obtain, $\mathbb{P}$-a.s., $Z_t = \mathbb{E}[Z_t \,|\, \mathcal{F}_t] = \mathbb{E}[\mathbf{1}_{\{t\le\tau^*\}} Z_t \,|\, \mathcal{F}_t] = h(t) \le 0$. ☐

Corollary 4.15 (Comparison principle)**.** *Let Assumptions 3.1 and 4.7 hold. Let $p \ge 1$ and let $u^{(1)} \in C_p(\Lambda)$ (resp. $u^{(2)} \in C_p(\Lambda)$) be a viscosity subsolution (resp. supersolution) to PPDE (4.2). If $u^{(1)}(T,\cdot) \le u^{(2)}(T,\cdot)$ on $\mathbb{W}$, then $u^{(1)} \le u^{(2)}$ on Λ.*

Proof. Let $(t,\mathbf{x}) \in \Lambda$. Set

$$g(r,\omega,z) \;:=\; F(r, X^{t,\mathbf{x}}(\omega), z + u^{(2)}(r, X^{t,\mathbf{x}}(\omega))) - F(r, X^{t,\mathbf{x}}(\omega), u^{(2)}(r, X^{t,\mathbf{x}}(\omega)))$$

and

$$Z_r(\omega) \;:=\; u^{(1)}(r, X^{t,\mathbf{x}}(\omega)) - u^{(2)}(r, X^{t,\mathbf{x}}(\omega)).$$

Due to Assumption 4.7, the map g satisfies the assumptions of Lemma 4.14. Moreover, by using the implication (iii) $\Rightarrow$ (ii) of Theorem 4.8 and the inequality $u^1(T,\cdot) - u^2(T,\cdot) \le 0$, we see that Z satisfies the assumption of Lemma 4.14. Then the claim follows as, $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$u^{(1)}(t, X^{t,\mathbf{x}}(\omega)) - u^{(2)}(t, X^{t,\mathbf{x}}(\omega)) \;=\; u^{(1)}(t,\mathbf{x}) - u^{(2)}(t,\mathbf{x}).$$ ☐

4.4 Existence and uniqueness

In this section we provide our main result. We shall consider the following assumption on the terminal condition ξ.

Assumption 4.16. *$\xi \in C(\mathbb{W};\mathbb{R})$ and, for some $C_\xi > 0$, $p \geq 1$,*

$$|\xi(\mathbf{x})| \leq C_\xi(1+\|\mathbf{x}\|_\infty^p), \qquad \forall\, \mathbf{x} \in \mathbb{W}. \tag{4.30}$$

Theorem 4.17. *Let Assumption 3.1 hold and let Assumptions 4.7, 4.16 hold with the same growth rate $p \geq 1$. Then PPDE (4.2) has a unique viscosity solution in the space $C_p(\Lambda)$ satisfying the terminal condition (4.3).*

Remark 4.18. *Uniqueness of viscosity solutions to PPDE (4.2) is already implied by the comparison principle in Corollary 4.15. However, it will be also a by-product of the fixed point argument used to prove the existence (Proposition 4.19).* □

Due to Theorem 4.8, the proof of the result above reduces to the study of the functional equation

$$u(t,\mathbf{x}) \;=\; \mathbb{E}\bigg[u(s,X^{t,\mathbf{x}}) + \int_t^s F(r,X^{t,\mathbf{x}},u(r,X^{t,\mathbf{x}}))dr\bigg], \;\; \text{for all } (t,\mathbf{x}) \in \Lambda,\ s \in [t,T], \tag{4.31}$$

with terminal condition

$$u(T,\cdot) = \xi(\cdot). \tag{4.32}$$

Existence and uniqueness of solutions to the functional equation (4.31)-(4.32) could be deduced from the theory of backward stochastic differential equations in Hilbert spaces (see Remark 4.21 below). However, for reader convenience, we provide here a direct proof that does not rely on the theory of BSDEs.

Proposition 4.19. *Let Assumption 3.1 hold and let Assumptions 4.7, 4.16 hold with the same growth rate $p \geq 1$. There exists a unique $\hat{u} \in C_p(\Lambda)$ solution to (4.31) with terminal condition (4.32).*

Proof. *Step I.* Fix a function $\zeta \in C_p(\Lambda)$, and let $0 \leq a \leq b \leq T$. Consider the nonlinear operator $\Gamma\colon C_p(\Lambda) \to C_p(\Lambda),\ u \mapsto \Gamma(u)$, defined by

$$\Gamma(u)(t,\mathbf{x}) := \mathbb{E}\left[\zeta(X^{t,\mathbf{x}}) + \mathbf{1}_{[a,b]}(t)\int_t^b F(s,X^{t,\mathbf{x}},u(s,X^{t,\mathbf{x}}))ds\right], \qquad \forall (t,\mathbf{x}) \in \Lambda. \tag{4.33}$$

First we note that actually Γ is well defined and maps $C_p(\Lambda)$ into itself: it follows from Assumption 4.7 and Corollary 3.9.

We now show that there exists $\varepsilon > 0$ such that, if $b-a < \varepsilon$, then Γ is a contraction on $C_p(\Lambda)$, hence admits a unique fixed point. Let $u,v \in C_p(\Lambda)$. Using Assumption 4.7(ii),

$$|\Gamma(u)(t,\mathbf{x}) - \Gamma(v)(t,\mathbf{x})| \;\;\leq\;\; \mathbb{E}\left[\mathbf{1}_{[a,b]}(t)\int_t^b \big|F(s,X^{t,\mathbf{x}},u(s,X^{t,\mathbf{x}})) - F(s,X^{t,\mathbf{x}},v(s,X^{t,\mathbf{x}})\big|\,ds\right]$$

$$
\begin{aligned}
&\le \hat{L}\,\mathbb{E}\left[\mathbf{1}_{[a,b]}(t)\int_t^b |u(s,X^{t,\mathbf{x}})-v(s,X^{t,\mathbf{x}})|ds\right]\\
&\le \hat{L}\,\|u-v\|_{C_p(\Lambda)}\,\mathbb{E}\left[\mathbf{1}_{[a,b]}(t)\int_t^b \left(1+\|X^{t,\mathbf{x}}\|_\infty^p\right)ds\right]\\
&\le \hat{L}\,\|u-v\|_{C_p(\Lambda)}\mathbf{1}_{[a,b]}(t)\int_t^b \left(1+M(1+\|\mathbf{x}\|_\infty^p)\right)ds\\
&\le \varepsilon\hat{L}\,(1+M)(1+\|\mathbf{x}\|_\infty^p)\,\|u-v\|_{C_p(\Lambda)}
\end{aligned}
$$

which yields

$$
\|\Gamma(u)-\Gamma(v)\|_{C_p(\Lambda)} \;\le\; \varepsilon\hat{L}(1+M)\|u-v\|_{C_p(\Lambda)}. \tag{4.34}
$$

Thus, Γ is a contraction whenever $\varepsilon<(\hat{L}(1+M))^{-1}$. For such ε, it admits a unique fixed point $\hat{u}$:

$$
\hat{u}(t,\mathbf{x}) \;=\; \mathbb{E}\left[\zeta(X^{t,\mathbf{x}})+\mathbf{1}_{[a,b]}(t)\int_t^b F(s,X^{t,\mathbf{x}},\hat{u}(s,X^{t,\mathbf{x}}))ds\right], \qquad \forall\,(t,\mathbf{x})\in\Lambda. \tag{4.35}
$$

Step II. We prove that, if a function $\hat{u}$ satisfies (4.35) for $(t,\mathbf{x})\in\Lambda$, $a\le t\le b$, then it also satisfies, for every $(t,\mathbf{x})\in\Lambda$ and every $(s,\mathbf{x})\in\Lambda$ with $a\le t\le s\le b$, the equality

$$
\hat{u}(t,\mathbf{x})=\mathbb{E}\left[\hat{u}(s,X^{t,\mathbf{x}})+\int_t^s F(s,X^{t,\mathbf{x}},\hat{u}(s,X^{t,\mathbf{x}}))ds\right], \tag{4.36}
$$

Indeed, using (3.10) and [1, Lemma 3.9, p. 55]

$$
\begin{aligned}
\hat{u}(s,X^{t,\mathbf{x}}) &= \mathbb{E}\left[\zeta(X^{s,\mathbf{y}})+\int_s^b F(r,X^{s,\mathbf{y}},\hat{u}(r,X^{s,\mathbf{y}}))dr\right]_{|\mathbf{y}=X^{t,\mathbf{x}}}\\
&= \mathbb{E}\left[\zeta(X^{s,X^{t,\mathbf{x}}})+\int_s^b F(r,X^{s,X^{t,\mathbf{x}}},\hat{u}(r,X^{s,X^{t,\mathbf{x}}}))dr\,\middle|\,\mathcal{F}_s\right]\\
&= \mathbb{E}\left[\zeta(X^{t,\mathbf{x}})+\int_s^b F(r,X^{t,\mathbf{x}},\hat{u}(r,X^{t,\mathbf{x}}))dr\,\middle|\,\mathcal{F}_s\right].
\end{aligned}
$$

Hence

$$
\mathbb{E}\left[\hat{u}(s,X^{t,\mathbf{x}})\right] \;=\; \mathbb{E}\left[\zeta(X^{t,\mathbf{x}})+\int_s^b F(r,X^{t,\mathbf{x}},\hat{u}(r,X^{t,\mathbf{x}}))dr\right]
$$

and we conclude by (4.35).

Step III. In this step we conclude the proof. Let a,b as in *Step I* and let us assume, without loss of generality, that $T/(b-a)=n\in\mathbb{N}$. By *Step I*, there exists a unique $\hat{u}_n\in C_p(\Lambda)$ satisfying

$$
\hat{u}_n(t,\mathbf{x}) \;:=\; \mathbb{E}\left[\xi(X^{t,\mathbf{x}})+\mathbf{1}_{[T-(b-a),T]}(t)\int_t^T F(s,X^{t,\mathbf{x}},\hat{u}_n(s,X^{t,\mathbf{x}}))ds\right], \qquad \forall(t,\mathbf{x})\in\Lambda.
$$

With a backward recursion argument, using *Step I*, we can find (uniquely determined) functions $\hat{u}_i\in C_p(\Lambda)$, $i=1,\dots,n$, such that

$$
\hat{u}_{i-1}(t,\mathbf{x}):=\mathbb{E}\left[\hat{u}_i(i(b-a),X^{t,\mathbf{x}})+\mathbf{1}_{[(i-1)(b-a),i(b-a)]}(t)\int_t^{i(b-a)} F(s,X^{t,\mathbf{x}},\hat{u}_i(s,X^{t,\mathbf{x}}))ds\right],
$$

for all $(t, \mathbf{x}) \in \Lambda$. Now define $\hat{u}(t,\cdot) = \sum_{1\leq i\leq n} \mathbf{1}_{[(i-1)(b-a),i(b-a))}(t)\hat{u}_i(t,\cdot) + \mathbf{1}_{\{T\}}(t)\xi(\cdot)$. To conclude the existence, we use recursively *Step II* to prove that $\hat{u}$ satisfies (4.31) with terminal condition (4.32).

Uniqueness follows from local uniqueness. Indeed, let $\hat{u}, \hat{v}$ be two solutions in $C_p(\Lambda)$ of (4.31)-(4.32) and define

$$T^* := \sup\left\{t \in [0,T]\colon \sup_{\mathbf{x}\in\mathbb{W}} |\hat{u}(t,\mathbf{x}) - \hat{v}(t,\mathbf{x})| > 0\right\},$$

with the convention $\sup\emptyset = 0$. By continuity of $\hat{u}, \hat{v}$, and since $\hat{u}(T,\cdot) = \hat{v}(T,\cdot)$, we have $\hat{u}(t,\cdot) \equiv \hat{v}(t,\cdot)$ for every $t \in [T^*, T]$. If $T^* = 0$, we have done. Assume, by contradiction, that $T^* > 0$. As done in *Step II*, one can prove that both $\hat{u}$ and $\hat{v}$ satisfy (4.36). In particular, if we consider the definition (4.33) with $\zeta(\cdot) = \hat{u}(T^*,\cdot) = \hat{v}(T^*,\cdot)$, $a = 0 \vee (T^* - \varepsilon)$, $b = T^*$, where $\varepsilon < (\hat{L}(1+M))^{-1}$, we have

$$\Gamma(\hat{u})(t,\mathbf{x}) = \hat{u}(t,\mathbf{x}) \quad \text{and} \quad \Gamma(\hat{v})(t,\mathbf{x}) = \hat{v}(t,\mathbf{x}), \qquad \forall\, (t,\mathbf{x}) \in \Lambda,\, \forall t \in [T^* - \varepsilon T^*].$$

Then, recalling (4.34), we get a contradiction and conclude. □

Remark 4.20. *If there exists a modulus of continuity w_F such that*

$$|F(t,\mathbf{x},y) - F(t',\mathbf{x}',y')| \;\leq\; w_F(\mathbf{d}_\infty((t,\mathbf{x}),(t',\mathbf{x}'))) + \hat{L}|y-y'|,$$

then Γ defined in (4.33) maps $UC(\Lambda)$ into itself. Hence, if ξ is uniformly continuous and the condition above on F holds, then the solution $\hat{u}$ belongs to $UC(\Lambda)$.

Remark 4.21 (Nonlinear Feynman-Kac formula for the function $\hat{u}$ in Proposition 4.19)**.** *Another way to solve the functional equation (4.31) is to consider the following backward stochastic differential equation*

$$Y_s \;=\; \xi(X^{t,\mathbf{x}}) + \int_s^T F(r, X^{t,\mathbf{x}}, Y_r)dr - \int_s^T Z_r dW_r, \qquad s \in [t,T]. \tag{4.37}$$

Then, it follows from Proposition 4.3 in [20] that, under Assumptions 3.1, 4.7, and 4.16 (with the same growth rate $p \geq 1$), for any $(t,\mathbf{x}) \in \Lambda$ there exists a unique solution $(Y_s^{t,\mathbf{x}}, Z_s^{t,\mathbf{x}})_{s\in[0,T]} \in \mathcal{H}^2_{\mathcal{P}}(\mathbb{R}) \times L^2_{\mathcal{P}}(H^*)$ to equation (4.37), which can be viewed as a Sobolev solution to PPDE (4.2) (see e.g. [2]). We also know that $Y_t^{t,\mathbf{x}}$ is constant, then we may define

$$\hat{u}(t,\mathbf{x}) \;:=\; Y_t^{t,\mathbf{x}} \;= \mathbb{E}\Big[\xi(X^{t,\mathbf{x}}) + \int_t^T F(s, X^{t,\mathbf{x}}, Y_s^{t,\mathbf{x}})ds\Big], \tag{4.38}$$

for all $(t,\mathbf{x}) \in \Lambda$. It can be shown, using the flow property of $X^{t,\mathbf{x}}$ and the uniqueness of the backward equation (4.37), that $Y_s^{t,\mathbf{x}} = \hat{u}(s, X^{t,\mathbf{x}})$ for all $s \in [t,T]$, $\mathbb{P}$-almost surely. Moreover, using the backward equation (4.37), the regularity of ξ and F, and the flow property of $X^{t,\mathbf{x}}$ with respect to $(t,\mathbf{x})$, we can prove that $\hat{u} \in C_p(\Lambda)$. This implies that $\hat{u}$ solves the functional equation (4.31) with terminal condition (4.32), and it is the same function of Proposition 4.19. Viceversa, we can also prove an existence and uniqueness result for the backward equation (4.37) if we know that there exists a unique *solution $\hat{u} \in C_p(\Lambda)$ to the functional equation (4.31) with terminal condition (4.32). In conclusion, $\hat{u}$ admits a nonlinear Feynman-Kac representation formula through a non-Markovian forward-backward stochastic differential equation given by:*

$$\begin{cases} X_s = e^{(s-t)A}\mathbf{x}_t + \int_t^s e^{(s-r)A} b(r,X)dr + \int_t^s e^{(s-r)A}\sigma(r,X)dW_r, & s \in [t,T], \\ X_s = \mathbf{x}_s, & s \in [0,t], \\ Y_s = \xi(X) + \int_s^T F(r,X,Y_r)dr - \int_s^T Z_r dW_r, & s \in [0,T]. \end{cases}$$

5 The Markovian case

In the Markovian case, i.e., when all data depend only on the present, infinite-dimensional PDEs of type (4.2)-(4.3) have been studied from the point of view of viscosity solutions starting from [26, 27, 28]. In this section we compare the results of the literature with the statement of our main Theorem 4.8 in this Markovian framework.

Hence, let us assume that the data b, σ, F, ξ satisfy all the assumptions used in the previous sections and, moreover, that they depend only on $x = \mathbf{x}_t$, instead of the whole path $\mathbf{x}$. The SDE (3.1) is no more path-dependent and takes the following form:

$$\begin{cases} dX_s = AX_s ds + b(s,X_s)ds + \sigma(s,X_s)dW_s, & s \in [t,T], \\ X_t = x \in H. \end{cases} \tag{5.1}$$

Accordingly, (1.1) becomes a non path-dependent ([7]) second order parabolic PDE in the Hilbert space H, which is formally written for $(t,x) \in [0,T) \times \mathcal{D}(A)$ as ([8])

$$-\partial_t u(t,x) - \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x)\sigma^*(t,x)D^2u(t,x)\right] - \langle Ax, Du(t,x)\rangle - \\ - \langle b(t,x), Du(t,x)\rangle - F(t,x,u(t,x)) = 0. \tag{5.2}$$

In such Markovian framework, the results of Section 4 still hold. Indeed, defining viscosity solutions of (5.2) as in Definition 4.3, with x in place of $\mathbf{x}$, we know from Theorem 4.17 that there exists a unique viscosity solution $\hat{u}$ to (5.2) and that it admits the probabilistic representation formula (4.38) of Remark 4.21, with x in place of $\mathbf{x}$.

On the other hand, equations like (5.2) have been studied in the literature, by means of what we call here the "standard" viscosity solution approach. This is performed, in the spirit of the finite-dimensional case, by computing the terms of (5.2) on smooth test functions suitably defined and using the method of doubling variables to prove the comparison. Such "standard" approach in infinite dimension has been first introduced in [26, 27, 28] and then developed in various papers (see e.g. [22, 23, 24, 25, 35]).

To compare our results with the ones obtained in the literature quoted above, we first introduce a concept of classical solution of (5.2).

[7]In this section we drop the final condition ξ. But it is important to notice that the PDE must be considered path-dependent even if only ξ depends on the past, while b, σ, F do not.

[8]Notice that the time derivative $\partial_t u(t,x)$ here appearing can denote equivalently the Dupire time-derivative defined in Appendix A or the standard partial right time-derivative, as in this Markovian case they coincide each other on $[0,T)$.

First of all, observe that (5.2) is well defined only in $[0,T)\times\mathcal{D}(A)$. In order to formally extend this set of definition we can consider the operator A^*, adjoint of A, defined on $\mathcal{D}(A^*)\subset H$, and express the term containing Ax in (5.2) by writing

$$\langle Ax, Du(t,x)\rangle = \langle x, A^*Du(t,x)\rangle,$$

which is well defined in $[0,T)\times H$ provided that $Du\in\mathcal{D}(A^*)$. Hence, to define classical solutions of such equation, we define the operator $\mathcal{L}_1$ as follows: the domain of definition of the solution is ($UC^{1,2}([0,T]\times H)$ denotes the space of maps $\psi\colon [0,T]\times H\to\mathbb{R}$ which are uniformly continuous together with their first time Fréchet derivative and their first and second spatial Fréchet derivatives)

$$\mathcal{D}(\mathcal{L}_1)=\Big\{\psi\in UC^{1,2}([0,T]\times H)\colon \text{ the maps } (t,x)\mapsto\langle x, A^*D\psi(t,x)\rangle,\ A^*D\psi(t,x),\\ \frac{1}{2}\mathrm{Tr}\left[\sigma(t,x)\sigma^*(t,x)D^2\psi(t,x)\right],\ \text{belong to } UC([0,T]\times H)\Big\},$$

and

$$\mathcal{L}_1\psi(t,x)=\partial_t\psi(t,x)+\frac{1}{2}\mathrm{Tr}\left[\sigma(t,x)\sigma^*(t,x)D^2\psi(t,x)\right]+\langle x, A^*D\psi(t,x)\rangle+\langle b(t,x), D\psi(t,x)\rangle.$$

Then we say that u is a classical solution of (5.2) if $u\in D(\mathcal{L}_1)$ and satisfies

$$-\mathcal{L}_1u(t,x)-F(t,x,u(t,x))=0,\qquad \forall(t,x)\in[0,T)\times H. \tag{5.3}$$

The standard definition of viscosity subsolution (supersolution) for (5.2) says roughly that, at any given $(t,x)\in[0,T)\times H$, the equation must be satisfied with $\le$ ($\ge$), when we substitute to the derivatives of $u(t,x)$ the derivatives of $\varphi(t,x)$, where φ is a suitably chosen test function.

Clearly, in this context test functions should be chosen in such a way that all terms of (5.2) have classical sense. Hence, their regularity must be substantially the one required for classical solutions, i.e., roughly, $\varphi\in\mathcal{D}(\mathcal{L}_1)$. This regularity is very demanding, much more than the one required in the finite dimensional case: requiring that $D\varphi\in\mathcal{D}(A^*)$ and the finite trace condition in the second order term strongly restricts the set of test functions. In this way the proof of the existence has not a greater structural difficulty with respect to the finite-dimensional case, but the uniqueness, which is based on a delicate construction of suitable test functions, becomes much harder.

To be more explicit, let us first give a definition of "naive" viscosity solution to (5.2).

Definition 5.1. (i) *An upper semicontinuous function $u\colon[0,T]\times H\to\mathbb{R}$ is called a **naive viscosity subsolution** of (5.2) if*

$$-\mathcal{L}_1\varphi(t,x)-F(t,x,u(t,x))\quad\le\quad 0,$$

for any $(t,x)\in[0,T)\times H$ and any function $\varphi\in D(\mathcal{L}_1)$ such that $\varphi-u$ has a local minimum at (t,x).

(ii) *A lower semicontinuous function* $u\colon [0,T] \times H \to \mathbb{R}$ *is called a* ***naive viscosity supersolution*** *of* (5.2) *if*

$$-\mathcal{L}_1\varphi(t,x) - F(t,x,u(t,x)) \;\; \geq \;\; 0,$$

for any $(t,x) \in [0,T) \times H$ *and any function* $\varphi \in D(\mathcal{L}_1)$ *such that* $\varphi - u$ *has a local maximum at* (t,x)*.*

(iii) *A continuous function* $u\colon [0,T] \times H \to \mathbb{R}$ *is called a* ***naive viscosity solution*** *of* (5.2) *if it is both a viscosity subsolution and a viscosity supersolution.*

If we adopt this definition, it is clear that the set of test functions used is strictly included in the one used in our Definition 4.3. Hence, if a function is a viscosity solution according to Definition 4.3, it must also be a viscosity solution according to Definition 5.1, while the opposite is, a priori, not true. Hence, if one were able to prove a uniqueness result for viscosity solution according to Definition 5.1, such a result would be more powerful than our existence and uniqueness Theorem 4.17. However, the technique used to prove uniqueness in finite dimension does not work with such a definition and there are no general uniqueness results with this definition.

In the literature concerning "standard" viscosity solutions in infinite dimension this problem has been overcome by introducing suitable restrictions on the family of equations and adding an ad hoc radial term g to each test function φ. We explain more in detail what is needed to apply such techniques to our equation (5.2); then we give a result obtained with such technique and compare it with our previous results.

To start, it is useful to rewrite equation (5.2) as follows:

$$-\partial_t u(t,x) - \langle x, A^* Du(t,x)\rangle - Lu(t,x) - F(t,x,u(t,x)) \;\; = \;\; 0, \quad \text{on } [0,T) \times H, \tag{5.4}$$

with, for any $u \in C^{1,2}([0,T] \times H)$ in the sense of Fréchet,

$$Lu(t,x) \;\; = \;\; \langle b(t,x), Du(t,x)\rangle + \frac{1}{2}\mathrm{Tr}\big[\sigma(t,x)\sigma^*(t,x)D^2u(t,x)\big].$$

To account for the "difficult" term $\langle x, A^* Du(t,x)\rangle$ we impose the following assumption.

Assumption 5.2. *The operator* A *is a maximal dissipative operator in* H*.*

Under Assumptions 3.1 and 5.2, it follows from [34] that there exists a symmetric, strictly positive, and bounded operator B on H such that A^*B is a bounded operator on H and

$$-A^*B + c_0 B \;\; \geq \;\; 0,$$

for some $c_0 > 0$.

Definition 5.3 (B-convergence, B-upper/-lower semicontinuity, B-continuity)**.** *Let* $\{x_n\}_{n\in\mathbb{N}} \subset H$ *be a sequence and let* $x \in H$*. We say that the sequence* $\{x_n\}_{n\in\mathbb{N}}$ *is* B*-convergent to* x*, if* $x_n \rightharpoonup x$ *and* $Bx_n \to Bx$ *in* H*.*

A function $u\colon [0,T] \times H \to \mathbb{R}$ *is said to be* B*-upper semicontinuous (resp.* B*-lower semicontinuous) if for any* $\{t_n\}_{n\in\mathbb{N}} \subset [0,T]$ *convergent to* $t \in [0,T]$*, and for any* $\{x_n\}_{n\in\mathbb{N}} \subset H$ B*-convergent to* $x \in H$*, we have*

$$\limsup_{n\to\infty} u(t_n,x_n) \leq u(t,x) \qquad (\textit{resp.}\;\; \liminf_{n\to\infty} u(t_n,x_n) \geq u(t,x)).$$

Finally, u *is* B*-continuous if it is* B*-upper and* B*-lower semicontinuous.*

We consider two classes of smooth (test) functions:

(C1) (the "smooth" part) $\varphi \in C^{1,2}([0,T] \times H)$, $D\varphi$ is $\mathcal{D}(A^*)$-valued, $\partial_t \varphi$, $A^* D\varphi$, and $D^2\varphi$ are uniformly continuous on $[0,T] \times H$, and φ is B-lower semiconinuous.

(C2) (the "radial" part) $g \in C^{1,2}([0,T] \times \mathbb{R})$ and, for every $t \in [0,T]$, the function $g(t,\cdot)$ is even on $\mathbb{R}$ and nondecreasing on $[0,\infty)$.

Definition 5.4. (i) *A B-upper semicontinuous function $u \colon [0,T] \times H \to \mathbb{R}$, which is bounded on bounded sets, is called a **viscosity subsolution** of (5.4) if*

$$-\partial_t(\varphi + g)(t,x) - \langle x, A^* D\varphi(t,x)\rangle - L(\varphi + g)(t,x) - F(t,x,u(t,x)) \quad \leq \quad 0,$$

for any $(t,x) \in [0,T) \times H$ and any pair of functions (φ, g) belonging, respectively, to the classes (C1)-(C2) above and such that $\varphi + g - u$ has a local minimum at (t,x).
(ii) *A B-lower semicontinuous function $u \colon [0,T] \times H \to \mathbb{R}$, which is bounded on bounded sets, is called a **viscosity supersolution** of (5.4) if*

$$-\partial_t(\varphi - g)(t,x) - \langle x, A^* D\varphi(t,x)\rangle - L(\varphi - g)(t,x) - F(t,x,u(t,x)) \quad \geq \quad 0,$$

for any $(t,x) \in [0,T) \times H$ and any pair of functions (φ, g) belonging, respectively, to the classes (C1)-(C2) above and such that $\varphi - g - u$ has a local maximum at (t,x).
(iii) *A function $u \colon [0,T] \times H \to \mathbb{R}$ is called a **viscosity solution** of (5.4) if it is both a viscosity subsolution and a viscosity supersolution.*

Remark 5.5. *The radial function g belonging to the class (C2) introduced in Definition 5.4 plays the role of cut-off function and is needed to produce, together with the B-continuity property, local/global minima and maxima of $\varphi + g - u$ and $\varphi - g - u$, respectively. However, the introduction of the radial function forces to impose Assumption 5.2 to get rid of the term $\langle Ax, Dg(t,x)\rangle$ which would come out from the gradient of g.*

Radial test functions could also be included in our Definition 4.3 when A is a maximal monotone operator without compromising the existence result (but note that it would be redundant including them in our definition, as they are not needed to prove uniqueness in Theorem 4.17). In this case, our Definition 4.3 would be stronger than Definition 5.4 in the sense that a viscosity subsolution (supersolution) in the sense of Definition 4.3 must be necessarily also a viscosity subsolution (supersolution) according to Definition 5.4. Indeed, a test function in the sense of Definition 5.4 would be also a test function in the sense of Definition 4.3. $\square$

We can now state a comparison theorem and an existence result for equation (5.4). Firstly, we need to introduce some notations. Let H_{-1} be the completion of H with respect to the norm

$$|x|_{-1}^2 \;:=\; \langle Bx, x\rangle.$$

Notice that H_{-1} is a Hilbert space with the inner product

$$\langle x, x'\rangle_{-1} \;:=\; \langle B^{1/2}x, B^{1/2}x'\rangle.$$

Let now $\{e_1, e_2, \dots\}$ be an orthonormal basis in H_{-1} made of elements of H. For $N > 2$ we denote $H_N = \mathrm{span}\{e_1, \dots, e_N\}$. Let $P_N \colon H_{-1} \to H_{-1}$ be the orthogonal projection onto H_N and denote $P_N^\perp = I - P_N$.

Theorem 5.6. *Let Assumptions 3.1, 4.30, 4.7, and 5.2 hold. In addition, let us impose the following assumptions.*

(i) *The map $y \mapsto F(t,x,y)$ is nonincreasing on $\mathbb{R}$, for any $(t,x) \in [0,T] \times H$.*

(ii) *There exists a positive constant $L_{b,\sigma}$ and a modulus of continuity $\omega_{\xi,F}$ such that*

$$\begin{aligned} |b(t,x) - b(t,x')| + \|\sigma(t,x) - \sigma(t,x')\|_2 \ &\leq \ L_{b,\sigma}|x - x'|_{-1}, \\ |\xi(x) - \xi(x')| + |F(t,x,y) - F(t,x',y)| \ &\leq \ \omega_{\xi,F}(|x - x'|_{-1}), \end{aligned}$$

for all $t \in [0,T]$, $x, x' \in H$, and $y \in \mathbb{R}$.

(iii) *$\sigma(t,x) \in L_2(H)$ for every $(t,x) \in \Lambda$ and the following limit holds*

$$\lim_{N\to\infty} \mathrm{Tr}\big[\sigma(t,x)\sigma^*(t,x)BP_N^\perp\big] \ = \ 0, \qquad \forall\,(t,x) \in [0,T] \times H.$$

Then, the following statements hold true.

(a) *Let u (resp. v) be a viscosity subsolution (resp. supersolution) to (5.4) satisfying a polynomial growth condition. If $u(T,\cdot) \leq v(T,\cdot)$, then $u \leq v$ on $[0,T] \times H$.*

(b) *Assume that $F = F(t,x)$ does not depend on y. Then, there exists a unique viscosity solution $\hat{u}$ of equation (5.4) satisfying the terminal condition $\hat{u}(T,\cdot) = \xi(\cdot)$ and it admits the probabilistic representation ([9])*

$$\hat{u}(t,x) \ = \ \mathbb{E}\left[\xi(X_T^{t,x}) + \int_t^T F(s, X_s^{t,x})ds\right], \qquad (t,x) \in [0,T] \times H.$$

Proof. See [35, Th. 3.2]. ([10]) ☐

Remark 5.7. *1. Notice that Assumption (i) of Theorem 5.6 is actually redundant in the framework of Assumption 4.7, due to the uniform Lipschitz property of F with respect to the last argument required therein. Indeed, let u (resp. v) be a viscosity subsolution (resp. supersolution) of (5.4) satisfying $u(T,\cdot) \leq \xi(\cdot)$ (resp. $v(T,\cdot) \geq \xi(\cdot)$). Our aim is to prove point (a) of Theorem 5.6, i.e., that $u \leq v$ on $[0,T] \times H$, without imposing Assumption (i) of the same theorem. To this end, set $\tilde{u}(t,x) := e^{\hat{L}t}u(t,x)$ and $\tilde{v}(t,x) := e^{\hat{L}t}v(t,x)$, for all $(t,x) \in [0,T] \times H$, where $\hat{L}$ is the constant in Assumption 4.7(ii). Then, by standard arguments (see, e.g., point (i) of Remark 3.9 in [13]), we can prove that $\tilde{u}$ (resp. $\tilde{v}$) is a viscosity subsolution (resp. supersolution) of (5.4) with $\tilde{F}(t,x,y) = -\hat{L}y + e^{\hat{L}t}F(t,x,e^{-\hat{L}t}y)$ in place of F. The Lipschitz property of F implies that the map $y \mapsto \tilde{F}(t,x,y)$ is nonincreasing, therefore we can apply point (a) of Theorem 5.6 to $\tilde{u}$ and $\tilde{v}$, which yields $\tilde{u} \leq \tilde{v}$ on $[0,T] \times H$. Then $u \leq v$ on $[0,T] \times H$ follows.*

[9]When H is finite dimensional, the probabilistic representation formula (4.38) provides the unique "standard" viscosity solution of (5.4) also when F depends on y, see [29].

[10]Actually, under the assumption that u, v are bounded in part (a), but this assumption can be relaxed to the polynomial growth case.

2. *Assumption (ii) in Theorem 5.6 is needed to exploit the B-continuity. Indeed the requirement of B-continuity on the sub(super)solutions is needed to generate maxima and minima in the proof of comparison. In this way one is obliged to assume these stronger conditions on the coefficients to ensure the existence of solutions (see [35]).*

3. *Assumption (iii) in Theorem 5.6 is needed since, to prove uniqueness, one has to use the so-called Ishii's Lemma which allows to perform the procedure of doubling variables. Up to now Ishii's Lemma is known to hold only in finite dimension, so the proof is performed through finite dimensional approximations: the condition (iii) ensures the convergence of such approximations.* □

We can conclude that, under the assumptions of Theorem 5.6(b), the two definitions of viscosity solution select the same solution. However, adopting our Definition 4.3 requires weaker assumptions to prove that the function $\hat{u}$ in (4.38) is the unique viscosity solution. In particular:

1. The map σ does not need to satisfy assumptions (iii) (which, in the constant σ case, would imply that $\sigma\sigma^*$ is a nuclear operator, hence reducing the applicability of the theory) as the proof of uniqueness does not require the use of Ishii's lemma on the corresponding finite-dimensional approximations.
2. The coefficients b, σ, F, and ξ do not need to be B-continuous with respect to x, as no local compactness is needed to produce local max/min in our sense.
3. The operator A does not need to be maximal monotone, as radial test functions are not needed to produce local max/min in our sense.

Roughly speaking, we can say that our definition allows to cover more general cases since the relation with the PDE is different in the following sense: the PDE is tested in analytical sense, but over test functions which satisfy the min/max condition only in a probabilistic sense and only when composed with the process $X^{t,x}$; indeed minimum (maximum) of $\varphi - u$ is not pointwise in a neighborhood of (t, x), but only in mean when composed with the process $X^{t,x}$.

Appendix

A Pathwise derivatives

The class of test functions used to define viscosity solutions for path-dependent PDEs has evolved from [13] and [14] to the recent work [33]. In Definition 4.1, which is inspired by [33], there is no more reference to the so-called pathwise (or functional, or Dupire) derivatives (for which we refer to [12] and also to [4, 5, 6, 7]), which are instead adopted in [13] and [14] (actually in [14] only the pathwise time derivative is used). This allows to go directly to the definition of viscosity solution, without pausing on the definition of pathwise derivatives, and, more generally, on recalling tools from functional Itô calculus. However, the class of test functions used in [13] or [14] has the advantage to be defined in a similar way to $C^{1,2}$, the standard class of smooth real-valued functions. In this case the object $\mathcal{L}u$ of (4.1), which in the present paper is only abstract, can be expressed in terms of the pathwise derivatives, as in the non path-dependent case, where $\mathcal{L}$ corresponds to a parabolic operator and can be written by means of time and spatial derivatives.

For this reason, in order to better understand Definition 4.1 and the notation $\mathcal{L}u$, we now define a subset of test functions $\mathscr{C}^{1,2}_X(\Lambda) \subset C^{1,2}_X(\Lambda)$ which admit the pathwise derivatives we are going to define. Here we follow [14], generalizing it to the present infinite dimensional setting.

Definition A.1. *Given $u \in C_p(\Lambda)$, for some $p \geq 1$, we define the* ***pathwise time derivative*** *of u at $(t, \mathbf{x}) \in \Lambda$ as follows:*

$$\begin{cases} \partial_t u(s, \mathbf{x}) := \lim_{h \to 0^+} \dfrac{u(s+h, \mathbf{x}_{\cdot \wedge s}) - u(s, \mathbf{x})}{h}, & s \in [0, T), \\ \\ \partial_t u(T, \mathbf{x}) := \lim_{s \to T^-} \partial_t u(s, \mathbf{x}), & s = T, \end{cases}$$

when these limits exist.

In the following definition A^* is the adjoint operator of A, defined on $\mathcal{D}(A^*) \subset H$.

Definition A.2. *Denote by $S(H)$ the Banach space of bounded and self-adjoint operators in the Hilbert space H endowed with the operator norm, and let $\mathcal{D}(A^*)$ be endowed with the graph norm, which renders it a Hilbert space. We say that $u \in C_p(\Lambda)$, for some $p \geq 1$, belongs to $\mathscr{C}^{1,2}_X(\Lambda)$ if:*

(i) *there exists $\partial_t u$ in Λ in the sense of Definition A.1 and it belongs to $C_p(\Lambda)$;*

(ii) *there exist two maps $\partial_{\mathbf{x}} u \in C_p(\Lambda; \mathcal{D}(A^*))$ and $\partial^2_{\mathbf{xx}} u \in C_p(\Lambda; S(H))$ such that $\mathrm{Tr}\left[\sigma\sigma^* \partial^2_{\mathbf{xx}} u\right] < +\infty$ in Λ and the following* ***functional Itô's formula*** *holds for all $(t, \mathbf{x}) \in \Lambda$ and $s \in [t, T]$:*

$$du(s, X^{t,\mathbf{x}}) \;=\; \mathcal{L}u(s, X^{t,\mathbf{x}})ds + \langle \sigma^*(s, X^{t,\mathbf{x}}) \partial_{\mathbf{x}} u(s, X^{t,\mathbf{x}}), dW_s \rangle, \tag{A.1}$$

where, for $(s, \mathbf{y}) \in \Lambda$,

$$\begin{aligned} \mathcal{L}u(s, \mathbf{y}) \;:=\; & \partial_t u(s, \mathbf{y}) + \langle \mathbf{y}_t, A^* \partial_{\mathbf{x}} u(s, \mathbf{y}) \rangle + \langle b(s, \mathbf{y}), \partial_{\mathbf{x}} u(s, \mathbf{y}) \rangle \\ & + \frac{1}{2} \mathrm{Tr}\big[\sigma(s, \mathbf{y}) \sigma^*(s, \mathbf{y}) \partial^2_{\mathbf{xx}} u(s, \mathbf{y})\big]. \end{aligned} \tag{A.2}$$

Given (i) above, we can call $\partial_{\mathbf{x}} u$ a ***pathwise first order spatial derivative*** *of u with respect to X and $\partial^2_{\mathbf{xx}} u$ a* ***pathwise second order spatial derivative*** *of u with respect to X and denote*

$$\partial^2_X u := \big\{ (\partial_{\mathbf{x}} u, \partial^2_{\mathbf{xx}} u) \in C_p(\Lambda; D(A^*)) \times C_p(\Lambda; S(H)) \colon \partial_{\mathbf{x}} u \text{ and } \partial^2_{\mathbf{xx}} u \text{ as in (ii)} \big\}. \qquad \square$$

Notice that, given $u \in \mathscr{C}^{1,2}_X(\Lambda)$ and $(t, \mathbf{x}) \in \Lambda$, the objects $\partial_{\mathbf{x}} u$ and $\partial^2_{\mathbf{xx}} u$ are not necessarily uniquely determined, while $\mathcal{L}u$ defined as in (A.2) and $\sigma^* \partial_{\mathbf{x}} u$ are uniquely determined. Indeed, this can be shown by identifying the finite variation part and the Brownian part in the functional Itô's formula (A.1). Moreover, (4.1) is satisfied with

$$\alpha(t,\mathbf{x}) = \partial_t u(t,\mathbf{x}) + \langle \mathbf{x}_t, A^*\partial_{\mathbf{x}} u(t,\mathbf{x})\rangle + \langle b(t,\mathbf{x}), \partial_{\mathbf{x}} u(t,\mathbf{x})\rangle + \frac{1}{2}\mathrm{Tr}\big[\sigma(t,\mathbf{x})\sigma^*(t,\mathbf{x})\partial^2_{\mathbf{x}\mathbf{x}} u(t,\mathbf{x})\big],$$

$$\beta(t,\mathbf{x}) = \sigma^*(t,\mathbf{x})\partial_{\mathbf{x}} u(t,\mathbf{x}).$$

In particular, $\mathscr{C}_X^{1,2}(\Lambda) \subset C_X^{1,2}(\Lambda)$ and the notation $\mathcal{L}u := \alpha$ introduced in Subsection 4.1 becomes clear.

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